

# ON EXTREMIZING SEQUENCES FOR THE ADJOINT RESTRICTION INEQUALITY ON THE CONE

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**ABSTRACT.** It is known that extremizers for the  $L^2$  to  $L^6$  adjoint Fourier restriction inequality on the cone in  $\mathbb{R}^3$  exist. Here we show that nonnegative extremizing sequences are precompact, after the application of symmetries of the cone. If we use the knowledge of the exact form of the extremizers, as found by Carneiro, then we can show that nonnegative extremizing sequences converge, after the application of symmetries.

## 1. INTRODUCTION

We study the properties of extremizing sequences for the Fourier restriction inequality on the cone in dimension 3 for which the adjoint restriction inequality can be rewritten equivalently as a convolution inequality. Carneiro [1], using the method developed by Foschi [6], found the exact form of the extremizers for the adjoint Fourier restriction inequalities in dimensions 3 and 4 but there seems to be no mention in the literature as to whether extremizing sequences are precompact after appropriate rescaling<sup>1</sup>. That is the question we try to answer in this paper using the methods developed by Christ and Shao [2] to analyze the corresponding inequality for the sphere in three dimensions.

We denote  $\Gamma^2 = \{(y, y') \in \mathbb{R}^2 \times \mathbb{R} : y' = |y|\}$ , the cone in  $\mathbb{R}^3$ . A function  $f$  on  $\Gamma^2$  can be identified, and we will do so, with a function from  $\mathbb{R}^2$  to  $\mathbb{R}$ . On  $\Gamma^2$  we consider the measure  $\sigma(y, y') = \delta(y' - |y|) \frac{dy dy'}{|y|}$ , that is, for a function  $f$  on the cone

$$\int_{\Gamma^2} f d\sigma = \int_{\mathbb{R}^2} f(y) \frac{dy}{|y|}.$$

We will denote the  $L^p(\Gamma^2, \sigma)$  norm of a function  $f$  as  $\|f\|_{L^p(\Gamma^2)}$ ,  $\|f\|_{L^p(\sigma)}$  or  $\|f\|_p$ .

The extension or adjoint Fourier restriction operator for the cone is given by

$$Tf(x, t) = \int_{\mathbb{R}^2} e^{ix \cdot y} e^{it|y|} f(y) |y|^{-1} dy \quad (1.1)$$

where  $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$  and  $f \in \mathcal{S}(\mathbb{R}^2)$ . With the Fourier transform  $\hat{g}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} g(x) dx$  we see that  $Tf(x, t) = \widehat{f\sigma}(-x, -t)$ .

A well known bound, [12], for  $Tf$  is given in the following theorem

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<sup>1</sup>[5] answers this question in the nonendpoint case and appeared while this manuscript was being prepared. We comment on that later in the introduction.

**Theorem 1.1.** *There exists  $C < \infty$  such that for all  $f \in L^2(\Gamma^2)$  the following inequality holds*

$$\|Tf\|_{L^6(\mathbb{R}^3)} \leq C\|f\|_{L^2(\Gamma^2)}. \quad (1.2)$$

Denote by  $\mathbf{C}$  the best constant in (1.2), that is

$$\mathbf{C} = \sup_{0 \neq f \in L^2(\Gamma^2)} \frac{\|Tf\|_{L^6(\mathbb{R}^3)}}{\|f\|_{L^2(\Gamma^2)}}. \quad (1.3)$$

The use of the Fourier transform allows us to write (1.2) in “convolution form”, namely

$$\begin{aligned} \|Tf\|_{L^6(\mathbb{R}^3)}^3 &= \|(Tf)^3\|_{L^2(\mathbb{R}^3)} = \|(\widehat{f\sigma})^3\|_{L^2(\mathbb{R}^3)} = \|(f\sigma * f\sigma * f\sigma)\|_{L^2(\mathbb{R}^3)} \\ &= (2\pi)^{3/2} \|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)}, \end{aligned} \quad (1.4)$$

thus  $\|T(f)\|_{L^6} \leq \|T(|f|)\|_{L^6}$ . This implies that if  $\{f_n\}_{n \in \mathbb{N}}$  is an extremizing sequence then so is  $\{|f_n|\}_{n \in \mathbb{N}}$ .

In what follows we will restrict attention to nonnegative functions  $f \in L^2(\Gamma^2)$ .

**Definition 1.2.** An extremizing sequence for the inequality (1.2) is a sequence  $\{f_n\}_{n \in \mathbb{N}}$  of functions in  $L^2(\Gamma^2)$  satisfying  $\|f_n\|_{L^2(\Gamma^2)} \leq 1$ , such that  $\|Tf_n\|_{L^6(\mathbb{R}^3)} \rightarrow \mathbf{C}$  as  $n \rightarrow \infty$ .

An extremizer for (1.2) is a function  $f \neq 0$  which satisfies  $\|Tf\|_{L^6(\mathbb{R}^3)} = \mathbf{C}\|f\|_{L^2}$ .

The main theorem of this paper is

**Theorem 1.3.** *Any extremizing sequence of nonnegative functions in  $L^2(\Gamma^2)$  for the inequality (1.2) is precompact up to symmetries, that is, every subsequence of an extremizing sequence has a sub-subsequence that converges in  $L^2(\Gamma^2)$  after the application of symmetries of the cone.*

The symmetries of the cone we refer to are dilations and Lorentz transformations that will be studied in Section 7, and Theorem 1.3 will be stated in a more precise form as Theorem 8.2 below.

With the knowledge of the exact form of the extremizers to (1.2) given by Carneiro in [1] one can improve Theorem 1.3 to obtain

**Theorem 1.4.** *Any extremizing sequence of nonnegative functions in  $L^2(\Gamma^2)$  for the inequality (1.2) converges in  $L^2(\Gamma^2)$ , after the application of symmetries of the cone.*

Define the function  $g$  by its Fourier transform as  $\hat{g}(y) = f(y)|y|^{-1}$ . Then

$$e^{it\sqrt{-\Delta}}g(x) := \frac{1}{(2\pi)^2} \int e^{ix \cdot y} e^{it|y|} \hat{g}(y) dy = \frac{1}{(2\pi)^2} Tf(x, t), \quad (1.5)$$

and

$$\|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} = \|f\|_{L^2(\Gamma^2)},$$

where we used the  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$  norm

$$\|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |\hat{g}(y)|^2 |y| dy.$$

We see that

$$(2\pi)^{-2} \|Tf\|_{L^6(\mathbb{R}^2)} \|f\|_{L^2(\Gamma^2)}^{-1} = \|e^{it\sqrt{-\Delta}}g\|_{L^6(\mathbb{R}^3)} \|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}^{-1}, \quad (1.6)$$

and (1.2) is equivalent to

$$\|e^{it\sqrt{-\Delta}}g\|_{L_{x,t}^6(\mathbb{R}^3)} \leq \frac{C}{(2\pi)^2} \|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}. \quad (1.7)$$

From (1.6),  $\{f_n\}_{n \in \mathbb{N}}$  is an extremizing sequence for (1.2) if and only if  $\{g_n\}_{n \in \mathbb{N}}$ , with  $\hat{g}_n(y) = f_n(y)|y|^{-1}$ , is an extremizing sequence for (1.7).

The problem of computing the best constant in (1.2) and the exact form of the extremizers was solved by Carneiro in [1]. With the normalization of the Fourier transform discussed earlier, Carneiro proves

**Theorem 1.5** ([1]). *For all  $f \in L^2(\Gamma^2)$ ,*

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^3)} \leq (2\pi)^{5/6} \|f\|_{L^2(\Gamma^2)}. \quad (1.8)$$

*and equality occurs in (1.8) if and only if  $f(y, |y|) = e^{-a|y|+b \cdot y+c}$ , where  $a, c \in \mathbb{C}$ ,  $b \in \mathbb{C}^2$ , and  $|\Re b| < \Re a$ .*

We will use this result to prove Theorem 1.4.

Fanelli, Vega and Visciglia proved in [5] a general existence theorem for extremizers of Strichartz inequalities. We state here the case of the cone, in its equivalent form via (1.5). For  $d \geq 2$  and  $0 \leq \sigma < \frac{d-1}{2}$  the following Strichartz estimates hold (see [5, Example 1.1])

$$\|e^{it\sqrt{-\Delta}}g\|_{L_{t,x}^{\frac{2(d+1)}{d-1-2\sigma}}(\mathbb{R}^{d+1})} \leq C \|g\|_{\dot{H}^{\frac{1}{2}+\sigma}(\mathbb{R}^d)}. \quad (1.9)$$

In [5], using “remodulation” (equation after [5, equation 2.12]) “rescaling” and “translation” ([5, equation 2.15]) the following theorem is proved,

**Theorem 1.6** ([5]). *Let  $d \geq 2$  and  $0 < \sigma < \frac{d-1}{2}$ . Then there exists an extremizer for (1.9). Moreover, extremizing sequences are precompact, after the application of symmetries: “remodulation”, “rescaling” and “translation”.*

We point out here that their method does not apply to the endpoint case studied in this paper,  $\sigma = 0$  and  $d = 2$ , because of the existence of further symmetries, Lorentz invariance, as discussed in Section 7. The symmetries referred to in Theorem 1.6, when expressed in the dual formulation for  $f \in L^2(\Gamma^2)$  are, in respective order:

- $f(y) \rightsquigarrow e^{is|y|}f(y)$ ,  $s \in \mathbb{R}$ ,
- $f(y) \rightsquigarrow \lambda^{1/2}f(\lambda y)$ ,  $\lambda > 0$  and
- $f(y) \rightsquigarrow e^{iy \cdot y_0}f(y)$ ,  $y_0 \in \mathbb{R}^2$ .

From the Lorentz invariance of inequality (1.2), and the fact that the Lorentz group is not generated modulo a compact subgroup by the elements listed above, it follows that the final conclusion of Theorem 1.6 cannot be true in the endpoint case  $d = 2$ ,  $\sigma = 0$ . This indicates that the proof in [5] likewise cannot apply to this endpoint case.

On the one hand, for  $d \geq 2$ , under admissibility conditions in  $(p, q)$  one has the Strichartz estimates

$$\|e^{it\sqrt{-\Delta}}g\|_{L_t^p L_x^q(\mathbb{R}^{d+1})} \leq C\|g\|_{\dot{H}^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}(\mathbb{R}^d)},$$

so that for the case of the  $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$  one needs  $p = q$  which then makes Theorem 1.1 in [5] not applicable.

On the other hand, at the level of the proof of [5, Theorem 1.1], one sees that [5, equation 2.12] does not hold for  $\sigma = 0$  (or  $s = 1/2$  as appears there) and  $d = 2$ . For this we show that there are extremizing sequences  $\{g_n\}_{n \in \mathbb{N}}$  such that  $\|e^{it\sqrt{-\Delta}}g_n\|_{L_t^\infty L_x^4} \rightarrow 0$  as  $n \rightarrow \infty$ .

This is the same as having extremizing sequence  $\{f_n\}_{n \in \mathbb{N}}$  such that  $\|Tf_n\|_{L_t^\infty L_x^4} \rightarrow 0$  as  $n \rightarrow \infty$ . For this we use the Lorentz invariance and the characterization of extremizers for the cone given in Theorem 1.5.

From Section 7,  $\|T(f \circ L)\|_{L^6(\mathbb{R}^3)} = \|Tf\|_{L^6(\mathbb{R}^3)}$ , for every Lorentz transformation  $L$  preserving  $\Gamma^2$ . Let  $f$  be an  $L^2$ -normalized extremizer, say  $f(x_1, x_2, x_3) = c_0 e^{-x_3}$ . We take a sequence of Lorentz transformations  $L^s$  and  $f \circ L^s$  is also an  $L^2$ -normalized extremizer. We now compute  $\|(Tf) \circ L^s\|_{L_t^\infty L_x^4}$ . We have

$$Tf(x, t) = \frac{2\pi c_0}{\sqrt{(1-it)^2 + |x|^2}},$$

and

$$|Tf(x, t)|^4 = \frac{(2\pi)^4 c_0^4}{(1-t^2 + |x|^2)^2 + 4t^2}.$$

Now we use  $L^s(x, t) = (\frac{x_1+st}{(1-s^2)^{1/2}}, x_2, \frac{t+sx_1}{(1-s^2)^{1/2}})$  and note that by making the change of variables  $u = (x_1 + st)(1-s^2)^{-1/2}$ ,  $v = x_2$  we obtain

$$\int |(Tf) \circ L^s(x, t)|^4 dx = (1-s^2)^{1/2} \int |Tf(x_1, x_2, sx_1 + t(1-s^2)^{1/2})|^4 dx.$$

Then, if  $s \neq 0$

$$\begin{aligned} \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^2} |(Tf) \circ L^s(x, t)|^4 dx &= (1-s^2)^{1/2} \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^2} |Tf(x_1, x_2, s(x_1 + t))|^4 dx \\ &= (2\pi)^4 c_0^4 (1-s^2)^{1/2} \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^2} \frac{dx_1 dx_2}{(1-s^2(x_1+t)^2 + x_1^2 + x_2^2)^2 + 4s^2(x_1+t)^2} \\ &= (2\pi)^4 c_0^4 (1-s^2)^{1/2} \sup_{t \in \mathbb{R}} \int_{\mathbb{R}^2} \frac{dx_1 dx_2}{(1-s^2x_1^2 + (x_1+t)^2 + x_2^2)^2 + 4s^2x_1^2}. \end{aligned}$$

It is not hard to show that for  $(s, t) \in [1/2, 1] \times \mathbb{R}$

$$\int_{\mathbb{R}^2} \frac{dx_1 dx_2}{(1-s^2x_1^2 + (x_1+t)^2 + x_2^2)^2 + 4s^2x_1^2} \leq C,$$

with  $C$  independent of  $s$  and  $t$ . Therefore

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}^2} |(Tf) \circ L^s(x, t)|^4 dx \leq C(1-s^2)^{1/2}.$$

Hence  $\lim_{s \rightarrow 1^-} \|T(f \circ L^s)\|_{L_t^\infty L_x^4} = 0$ .

**Notation:** We will write  $X \lesssim Y$  or  $Y \gtrsim X$  to denote an estimate of the form  $X \leq CY$ , and  $X \asymp Y$  to denote an estimate of the form  $cY \leq X \leq CY$ , where  $0 < c, C < \infty$  are constants depending on fixed parameters of the problem, but independent of  $X$  and  $Y$ .

When writing integrals, we will sometimes drop the domain of integration or the measure when it is clear from context.

## 2. THE STRUCTURE OF THE PAPER AND THE IDEA OF THE PROOF

The proof of Theorem 1.3 follows the lines of the proof of precompactness of extremizing sequences for the adjoint Fourier operator on the sphere  $S^2 \subset \mathbb{R}^3$  given in [2].

In Section 3 we give a (known [12], [10, Chapter 2]) proof of Theorem 1.1, with a view towards a refinement in terms of a cap space, as used in [2] and proved in [4] for compact surfaces in  $\mathbb{R}^3$  of nonvanishing Gaussian curvature. In Section 4 we obtain bounds that we will use in Section 5 to obtain the following cap estimate,

$$\|Tf\|_{L^6(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\Gamma^2)}^{1-\gamma/2} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} d\sigma \right)^{\gamma/3}, \quad (2.1)$$

where the supremum ranges over all “caps”  $\mathcal{C} \subset \Gamma^2$  and  $\gamma > 0$  is a small universal constant. This is the analog of Lemma 6.1 in [2].

For a function satisfying  $\|Tf\|_{L^6(\mathbb{R}^3)} \geq \delta \mathbf{C} \|f\|_{L^2(\Gamma^2)}$ , the estimate in (2.1) allows the extraction of a cap  $\mathcal{C}$  with good properties:  $f$  can be decomposed as the sum of two functions with disjoint support  $f = g + h$  and  $g$ , which is comparable to  $f\chi_{\mathcal{C}}$ , satisfies

$$|g(x)| \leq C_\delta \|f\|_2 |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x), \text{ and } \|g\|_2 \geq \eta_\delta \|f\|_2.$$

This is the content of Section 6. In Section 7 we discuss symmetries of the cone. This includes dilations and Lorentz transformations and they allow us to take a cap  $\mathcal{C}$  and transform it into a cap  $\mathcal{C}'$  with better properties:  $\mathcal{C}'$  is contained in a bounded region, independent of the extremizing sequence, and has big measure.

The existence of symmetries of  $(\Gamma^2, \sigma)$  simplifies the argument, compared to [2]. Two ways are possible, use the arguments of Fanelli, Vega and Visciglia contained in [4] and [5] carried out in Section 8; or use the decomposition algorithm as done by Christ and Shao and carried out in Section 9.

For the argument based on [4] and [5], a single extraction of a cap and the use of symmetries is enough to prove precompactness. In the case of the argument based on [2], a cap decomposition is needed. For an extremizing sequence, the cap decomposition is used to show that after dilations and Lorentz transformations, the extremizing sequence has a uniform  $L^2$ -decay at infinity. The uniform decay plus a result inspired from [4] allows us to complete the proof of precompactness.

In the last section, we prove that extremizing sequences converge, after the application of symmetries of the cone. This is an easy task, that follows from the fact that

the extremizers for (1.2) are known and that the group of symmetries of the cone acts transitively in the set of extremizers.

### 3. THE ADJOINT FOURIER RESTRICTION INEQUALITY

Abusing notation we will write  $f(r, \theta) = f(x)$ , where  $x = (r \cos \theta, r \sin \theta)$ , that is the polar representation of  $x$ . Note that in polar coordinates the measure  $|y|^{-1} dy$  becomes  $dr d\theta$ .

In the proof of Theorem 1.1 we will need the following standard lemma.

**Lemma 3.1** (Fractional integration). *Let  $1 < p, q < \infty$ . Then for any  $g \in L^p(\mathbb{R})$ ,  $h \in L^q(\mathbb{R})$  the following holds*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |g(s)h(t)| |t-s|^{-\alpha} ds dt \leq C_{p,q} \|g\|_{L^p(\mathbb{R})} \|h\|_{L^q(\mathbb{R})},$$

where  $\alpha = 2 - \frac{1}{p} - \frac{1}{q}$  and  $\frac{1}{p} + \frac{1}{q} > 1$ .

From Lemma 3.1 we have

**Lemma 3.2.** *Let  $1 < p, q < \infty$  with  $\frac{1}{p} + \frac{1}{q} > 1$  and let  $\alpha = 2 - \frac{1}{p} - \frac{1}{q}$ . Then for any  $g \in L^p([0, 2\pi])$ ,  $h \in L^q([0, 2\pi])$  the following holds*

$$\int_0^{2\pi} \int_0^{2\pi} |g(s)h(t)| |\sin(t-s)|^{-\alpha} ds dt \leq C_{p,q} \|g\|_{L^p} \|h\|_{L^q}. \quad (3.1)$$

*Proof.* We split the integral in sixteen pieces according to  $[0, 2\pi] = [0, \pi/2] \cup [\pi/2, \pi] \cup [\pi, 3\pi/2] \cup [3\pi/2, 2\pi]$ , and then it will be enough to show that

$$\int_{m\pi/2}^{(m+1)\pi/2} \int_{n\pi/2}^{(n+1)\pi/2} |g(s)h(t)| |\sin(t-s)|^{-\alpha} ds dt \leq C_{p,q} \|g\|_{L^p} \|h\|_{L^q},$$

for all  $m, n \in \{0, 1, 2, 3\}$ . For this we use a simple change of variable that allows us to use Lemma 3.1.

If  $t, s \in [j\pi/2, (j+1)\pi/2]$ , for some  $j \in \{0, 1, 2, 3\}$ , then  $|t-s| \leq \pi/2$  and we use that  $\frac{2}{\pi}|t-s| \leq |\sin(t-s)| \leq |t-s|$ .

If  $s \in [0, \pi/2]$  and  $t \in [\pi, 3\pi/2]$  we can use the change of variables  $t' = t - \pi$  so that  $t' \in [0, \pi/2]$ . We note that  $|\sin(t-s)| = |\sin(t'-s)|$ .

If  $s \in [0, \pi/2]$  and  $t \in [\pi/2, \pi]$  we further split the intervals as  $[0, \pi/2] = [0, \pi/4] \cup [\pi/4, \pi/2]$  and  $[\pi/2, \pi] = [\pi/2, 3\pi/4] \cup [3\pi/4, \pi]$ . If  $s \in [0, \pi/4]$  and  $t \in [\pi/2, 3\pi/4]$  or if  $s \in [\pi/4, \pi/2]$  and  $t \in [3\pi/4, \pi]$ , then  $|\sin(t-s)| \geq 1/\sqrt{2}$  and the desired inequality follows from an application of Hölder's inequality. If  $s \in [\pi/4, \pi/2]$  and  $t \in [\pi/2, 3\pi/4]$ , then  $|t-s| \leq \pi/2$  and we can use the inequality  $\frac{2}{\pi}|t-s| \leq |\sin(t-s)| \leq |t-s|$  as in the first case discussed. Finally, if  $s \in [0, \pi/4]$  and  $t \in [3\pi/4, \pi]$  we use the substitution  $t' = t - \pi$  so that  $t' \in [-\pi/4, 0]$ . Since  $|\sin(t-s)| = |\sin(t'-s)|$  and  $|t'-s| \leq \pi/2$  we can conclude as before.

The other cases follow in the same way.  $\square$

*Proof of Theorem 1.1.* We split  $f(y) = \sum_{k \in \mathbb{Z}} f_k(y)$  where  $f_k(y) = f(y)\chi_{2^{k-1} \leq |y| < 2^k}$ . Then

$$(Tf)^2(x, t) = \sum_{k, k' \in \mathbb{Z}} Tf_k \cdot Tf_{k'}.$$

Taking  $L^3$  norm in both sides, using the triangle inequality and Lemma 3.3 below we get

$$\|Tf\|_{L^6}^2 \leq C \sum_{k, k'} 2^{-|k-k'|/6} \|f_k\|_{L^2(\sigma)} \|f_{k'}\|_{L^2(\sigma)}.$$

To conclude we use the Cauchy-Schwarz inequality

$$\|Tf\|_{L^6}^2 \leq C \left( \sum_{k, k'} 2^{-|k-k'|/6} \|f_k\|_{L^2(\sigma)}^2 \right)^{1/2} \left( \sum_{k, k'} 2^{-|k-k'|/6} \|f_{k'}\|_{L^2(\sigma)}^2 \right)^{1/2} \leq C \|f\|_{L^2(\sigma)}^2. \quad \square$$

**Lemma 3.3.** *There exists a constant  $C < \infty$  with the following property. Let  $k, k' \in \mathbb{Z}$  and  $f, g \in L^2(\Gamma^2)$  with  $f$  and  $g$  supported in the regions  $2^{k-1} \leq |y| < 2^k$  and  $2^{k'-1} \leq |y| < 2^{k'}$  respectively, then*

$$\|Tf \cdot Tg\|_{L^3(\mathbb{R}^3)} \leq C 2^{-|k-k'|/6} \|f\|_{L^2(\Gamma^2)} \|g\|_{L^2(\Gamma^2)}. \quad (3.2)$$

*Proof.* We can split

$$f(r, \theta)g(r', \theta') = f(r, \theta)g(r', \theta')(\chi_{r > r'} + \chi_{r < r'}) (\chi_{\theta > \theta'} + \chi_{\theta < \theta'}) \text{ for a.e. } (r, r', \theta, \theta').$$

Thus by the triangle inequality we can assume, without loss of generality, that  $\theta > \theta'$  and  $r < r'$  in the support of  $f(r, \theta)g(r', \theta')$ .

Using polar coordinates and Fubini's Theorem we have

$$\begin{aligned} Tf \cdot Tg(x, t) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{ix \cdot (y+y')} e^{it(|y|+|y'|)} f(y)g(y') |y|^{-1} |y'|^{-1} dy dy' \\ &= \int e^{ix \cdot (r \cos \theta + r' \cos \theta', r \sin \theta + r' \sin \theta')} e^{it(r+r')} f(r, \theta)g(r', \theta') d\theta d\theta' dr dr'. \end{aligned}$$

We make the following change of variables

$$(r, r', \theta, \theta') \mapsto (u, s, \varrho) = (r \cos \theta + r' \cos \theta', r \sin \theta + r' \sin \theta', r + r', r),$$

which is injective in the region where  $\theta > \theta'$ ,  $r < r'$ . The Jacobian of the transformation is

$$J^{-1} = \frac{\partial(u, s, \varrho)}{\partial(r, r', \theta, \theta')} = rr' \sin(\theta - \theta').$$

Using the change of variables

$$Tf \cdot Tg(x, t) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^3} e^{ix \cdot u} e^{its} f(y)g(y') J du ds \right) d\varrho,$$

and by Minkowski's inequality and Hausdorff-Young inequality,

$$\begin{aligned} \|Tf \cdot Tg\|_{L^3} &\leq \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^3} e^{ix \cdot u} e^{its} f(y) g(y') J du ds \right\|_{L^3} d\varrho \\ &\leq C \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y) g(y') J|^{3/2} du ds \right)^{2/3} d\varrho \\ &= C \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y) g(y')|^{3/2} (rr')^{-1/2} |\sin(\theta - \theta')|^{-1/2} J du ds \right)^{2/3} d\varrho. \end{aligned}$$

We now use that  $r \asymp 2^k$ ,  $r' \asymp 2^{k'}$  and Hölder's inequality to obtain

$$\begin{aligned} \|Tf \cdot Tg\|_{L^3} &\leq C(2^k 2^{k'})^{-1/3} (2^k)^{1/3} \left( \int |f(y) g(y')|^{3/2} |\sin(\theta - \theta')|^{-1/2} J du ds d\varrho \right)^{2/3} \\ &= C(2^k 2^{k'})^{-1/3} (2^k)^{1/3} \left( \int |f(y) g(y')|^{3/2} |\sin(\theta - \theta')|^{-1/2} d\theta d\theta' dr dr' \right)^{2/3}. \end{aligned} \quad (3.3)$$

On the other hand, by Lemma 3.2

$$\begin{aligned} &\int |f(y) g(y')|^{3/2} |\sin(\theta - \theta')|^{-1/2} d\theta d\theta' dr dr' \\ &\leq C \int \left( \int |f(r, \theta)|^2 d\theta \right)^{3/4} dr \cdot \int \left( \int |g(r', \theta')|^2 d\theta' \right)^{3/4} dr' \\ &\leq C(2^k 2^{k'})^{1/4} \left( \int |f(r, \theta)|^2 dr d\theta \right)^{3/4} \left( \int |g(r', \theta')|^2 dr' d\theta' \right)^{3/4}. \end{aligned}$$

Then, as  $2^k \leq 2^{k'}$

$$\begin{aligned} \|Tf \cdot Tg\|_{L^3} &\leq C(2^k 2^{k'})^{-1/3} \min((2^k)^{1/3}, (2^{k'})^{1/3}) (2^k 2^{k'})^{1/6} \|f\|_{L^2_{r, \theta}} \|g\|_{L^2_{r, \theta}} \\ &= C 2^{-(k+k')/6} \min((2^k)^{1/3}, (2^{k'})^{1/3}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}. \end{aligned}$$

We note that  $2^{-(k+k')/6} \min((2^k)^{1/3}, (2^{k'})^{1/3}) = 2^{-|k-k'|/6}$ , so

$$\|Tf \cdot Tg\|_{L^3} \leq C 2^{-|k-k'|/6} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}. \quad \square$$

**Proposition 3.4.** *There exists a constant  $C < \infty$  with the following property. Let  $f \in L^2(\Gamma^2)$  and for  $k \in \mathbb{Z}$  let  $f_k(y) = f(y) \chi_{\{2^{k-1} \leq |y| < 2^k\}}$ . Then*

$$\|Tf\|_{L^6(\mathbb{R}^3)} \leq C \left( \sum_{k \in \mathbb{Z}} \|f_k\|_{L^2(\Gamma^2)}^3 \right)^{1/3}.$$

*Proof.* By rewriting  $\|Tf\|_{L^6(\mathbb{R}^3)}^3$  as the  $L^2$  norm of a trilinear form and using the triangle inequality we have

$$\|Tf\|_{L^6}^3 = \|Tf \cdot Tf \cdot Tf\|_{L^2} = \left\| \sum_{i,j,k} Tf_i \cdot Tf_j \cdot Tf_k \right\|_{L^2} \leq \sum_{i,j,k} \|Tf_i \cdot Tf_j \cdot Tf_k\|_{L^2}.$$

Now for each  $i, j, k$ , without loss of generality we can assume that  $|j - k| = \max(|i' - j'| : i', j' \in \{i, j, k\})$ . Using Hölder's inequality, Theorem 1.1 and Lemma 3.3 we get

$$\|Tf_i \cdot Tf_j \cdot Tf_k\|_{L^2} \leq \|Tf_i\|_{L^6} \|Tf_j \cdot Tf_k\|_{L^3} \leq C 2^{-|j-k|/6} \|f_i\|_{L^2} \|f_j\|_{L^2} \|f_k\|_{L^2}. \quad (3.4)$$



Now, using the maximality of  $|j - k|$  we see that  $|j - k| \geq \frac{1}{3}|i - j| + \frac{1}{3}|j - k| + \frac{1}{3}|k - i|$ , and hence from (3.4),

$$\|Tf_i \cdot Tf_j \cdot Tf_k\|_{L^2} \leq 2^{-|i-j|/18} 2^{-|j-k|/18} 2^{-|k-i|/18} \|f_i\|_{L^2} \|f_j\|_{L^2} \|f_k\|_{L^2}.$$

Then

$$\|Tf\|_{L^6}^3 \leq C \sum_{i,j,k} 2^{-|i-j|/18} 2^{-|j-k|/18} 2^{-|k-i|/18} \|f_i\|_{L^2} \|f_j\|_{L^2} \|f_k\|_{L^2},$$

and a final application of Hölder's inequality gives the desired conclusion

$$\|Tf\|_{L^6}^3 \leq C \sum_{i,j,k} 2^{-|i-j|/18} 2^{-|j-k|/18} 2^{-|k-i|/18} \|f_k\|_{L^2}^3 \leq C \sum_{k \in \mathbb{Z}} \|f_k\|_{L^2}^3. \quad \square$$

#### 4. PRELIMINARIES FOR THE CAP BOUND FOR THE ADJOINT FOURIER OPERATOR

Recall that in the computation of  $\|(Tf)^2\|_{L^3}$ , in equation (3.3) with  $g = f$ , we came across the expression

$$\int |f(r, \theta) f(r', \theta')|^{3/2} |\sin(\theta - \theta')|^{-1/2} d\theta d\theta' dr dr'.$$

By assuming the angular support of  $f$  is contained in the region  $0 \leq \theta \leq \frac{\pi}{2}$ , that is  $f(r, \theta) = 0$  if  $\theta \notin [0, \frac{\pi}{2}]$ , we can study instead the comparable expression

$$\int |f(r, \theta) f(r', \theta')|^{3/2} |\theta - \theta'|^{-1/2} d\theta d\theta' dr dr'.$$

Instead of using fractional integration in  $\theta, \theta'$  and Hölder's inequality in  $r, r'$  we want to obtain a “cap type” inequality for  $T$  of the form in Theorem 4.2 in [7].

**Definition 4.1.** By a cap  $\mathcal{C}$  we mean a set  $\mathcal{C} \subset \Gamma$  whose projection to the plane  $\mathbb{R}^2 \times \{0\}$  is of the form  $[2^{k-1}, 2^k] \times J$ , when written in polar coordinates  $(r, \theta)$ , where  $k \in \mathbb{Z}$  and  $J \subset [0, 2\pi]$  is an interval. We will identify the cap  $\mathcal{C}$  with its projection to the  $xy$ -plane and write  $\mathcal{C} = [2^{k-1}, 2^k] \times J$ .

For a cap  $\mathcal{C} = [2^{k-1}, 2^k] \times J$ ,  $|\mathcal{C}| := \sigma(\mathcal{C}) = 2^{k-1}|J|$ , and for any  $\lambda \geq 0$ ,  $\lambda\mathcal{C} = [\lambda 2^{k-1}, \lambda 2^k] \times J$ , so  $\sigma(\lambda\mathcal{C}) = \lambda\sigma(\mathcal{C})$ .

**Definition 4.2.** Let  $0 < \alpha < 1$  and  $p = 2/(2 - \alpha)$ . Define, for  $f, g \in L^p(\mathbb{R})$ , the bilinear operator

$$B(f, g) = \int_{\mathbb{R}^2} f(x) g(x') |x - x'|^{-\alpha} dx dx'. \quad (4.1)$$

Note that the kernel  $x \in \mathbb{R} \mapsto |x|^{-\alpha}$  has a strictly positive Fourier transform and thus  $B$  is nondegenerate and satisfies the Cauchy-Schwarz inequality  $|B(f, g)|^2 \leq B(f, f) B(g, g)$ .

Lemma 3.1 implies that  $|B(f, f)| \leq C_p \|f\|_{L^p(\mathbb{R})}^2$ . We can say more if we work with the Lorentz spaces  $L^{p,q}(\mathbb{R})$  (see [11] for an introduction to Lorentz spaces). We have the following bound for  $B$  [8]

$$|B(f, f)| \lesssim \|f\|_{L^{p,2}(\mathbb{R})}^2.$$

This bound will allow us to prove the following

**Proposition 4.3.** *Let  $0 < \alpha < 1$  and  $p = 2/(2 - \alpha)$ . There exist constants  $C < \infty$  and  $\delta \in (0, 2)$  such that for all  $f \in L^p(\mathbb{R})$  the following inequality holds,*

$$B(f, f) \leq C \|f\|_{L^p(\mathbb{R})}^{2-\delta} \sup_{k, I} \|f_k\|_{L^p(\mathbb{R})}^\delta \left( \frac{|E_k \cap I|}{|E_k| + |I|} \right)^\delta,$$

where  $I$  ranges over all compact intervals of  $\mathbb{R}$ ,  $E_k = \{x \in \mathbb{R} : 2^k \leq |f(x)| < 2^{k+1}\}$  and  $f_k = f \chi_{E_k}$ ,  $k \in \mathbb{Z}$ .

*Proof.* We will use the following characterization of the  $L^{p,2}$  norm. If we decompose  $f$  as in the statement of the proposition,  $f = \sum_{k \in \mathbb{Z}} f_k$  where  $f_k$  have disjoint supports,  $E_k$ , and  $2^k \chi_{E_k} \leq |f_k| < 2^{k+1} \chi_{E_k}$ , then

$$\|f\|_{L^{p,2}(\mathbb{R})}^2 \asymp \sum_{k \in \mathbb{Z}} \|f_k\|_{L^p(\mathbb{R})}^2. \quad (4.2)$$

It follows from (4.2) that  $\|f\|_{L^{p,2}}^2 \lesssim \|f\|_{L^p}^p \sup_k \|f_k\|_{L^p}^{2-p}$ , from where the following bound is obtained

$$|B(f, f)| \lesssim \|f\|_{L^p(\mathbb{R})}^p \sup_{k \in \mathbb{Z}} \|f_k\|_{L^p(\mathbb{R})}^{2-p}.$$

We can improve the previous estimate. For this, let  $\eta > 0$ ,  $S = \{k : \|f_k\|_p \geq \eta \|f\|_p\}$ , and  $g = \sum_{k \in S} f_k$ . Then  $|B(f - g, f - g)| \lesssim \eta^{2-p} \|f\|_{L^p}^2$ . Since  $\|f\|_{L^p}^p = \sum_k \|f_k\|_{L^p}^p$  we obtain that  $|S| \leq \eta^{-p}$ . Therefore, by Cauchy-Schwarz

$$|B(g, g)|^{1/2} \leq \sum_{k \in S} |B(f_k, f_k)|^{1/2} \leq |S| \max_{k \in S} |B(f_k, f_k)|^{1/2} \leq \eta^{-p} \max_{k \in S} |B(f_k, f_k)|^{1/2}.$$

We deduce that

$$|B(f, f)|^{1/2} \leq |B(f - g, f - g)|^{1/2} + |B(g, g)|^{1/2} \leq \eta^{(2-p)/2} \|f\|_{L^p} + \eta^{-p} \max_{k \in S} |B(f_k, f_k)|^{1/2},$$

and squaring we obtain that for all  $\eta > 0$

$$|B(f, f)| \lesssim \eta^{2-p} \|f\|_{L^p}^2 + \eta^{-2p} \sup_{k \in \mathbb{Z}} |B(f_k, f_k)|.$$

Optimizing in  $\eta$  gives

$$|B(f, f)| \lesssim \sup_{k \in \mathbb{Z}} |B(f_k, f_k)|^{\delta/2} \|f\|_{L^p}^{2-\delta}, \quad (4.3)$$

for some  $\delta \in (0, 1)$  (the optimization gives  $\delta = 2(2 - p)/(2 + p)$ ). Thus it is then enough to obtain a bound on  $B(f, f)$  where  $f = \chi_E$ .

**Lemma 4.4.** *There exist  $C < \infty$  and  $\gamma \in (0, 1)$  with the following property. For every  $E$  subset of  $\mathbb{R}$  of finite measure*

$$B(\chi_E, \chi_E) \leq C \|\chi_E\|_{L^p(\mathbb{R})}^2 \left( \sup_I \frac{|E \cap I|}{|E| + |I|} \right)^\gamma, \quad (4.4)$$

where the supremum ranges over all compact intervals  $I$  of  $\mathbb{R}$ .

*Proof.* Let  $\{I_j^k\}_{j \in \mathbb{Z}}$  be a partition of the real line into intervals of equal length  $2^k$ . Then

$$\begin{aligned} B(\chi_E, \chi_E) &= \iint \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} dx dy = \sum_k \iint_{\{2^{k-1} \leq |x-y| < 2^k\}} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} dx dy \\ &\asymp \sum_k \sum_j 2^{-k\alpha} |E \cap I_j^k| |E \cap \tilde{I}_j^k| \\ &\lesssim \sum_k \sum_j 2^{-k\alpha} |E \cap \tilde{I}_j^k|^2 \end{aligned}$$

where  $\tilde{I}_j^k$  has the same center as  $I_j^k$  and double length. From now on we will rename  $\tilde{I}_j^k$  by  $I_j^k$ .

Now we fix  $k$  and estimate  $\sum_j 2^{-k\alpha} |E \cap I_j^k|^2$ . Let  $n$  be such that  $2^n \leq |E| < 2^{n+1}$ . We will divide the analysis into the cases where  $k \leq n$  and  $k > n$ . Recall that  $p = 2/(2-\alpha)$ , and let  $\gamma \in (0, 1)$  be a number to be determined later. We first consider the case  $k \leq n$ . We have

$$\begin{aligned} \sum_j 2^{-k\alpha} |E \cap I_j^k|^2 &\leq \sum_j |E \cap I_j^k| 2^{-k\alpha} \sup_i |E \cap I_i^k| \\ &\lesssim |E| 2^{-k\alpha} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma 2^{k(1-\gamma)} |E|^\gamma \\ &\asymp |E|^{1+\gamma} 2^{k(1-\alpha-\gamma)} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma \\ &= |E|^{2-\alpha} |E|^{-1+\alpha+\gamma} 2^{k(1-\alpha-\gamma)} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma \\ &\lesssim \|\chi_E\|_{L^p}^2 2^{-(n-k)(1-\alpha-\gamma)} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma. \end{aligned}$$

Now if  $k > n$  we will have

$$\begin{aligned} \sum_j 2^{-k\alpha} |E \cap I_j^k|^2 &\leq \sum_j |E \cap I_j^k| 2^{-k\alpha} \sup_i |E \cap I_i^k| \\ &\lesssim |E| 2^{-k\alpha} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma 2^{k\gamma} |E|^{1-\gamma} \\ &\asymp |E|^{2-\gamma} 2^{-k(\alpha-\gamma)} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma \\ &= |E|^{2-\alpha} |E|^{\alpha-\gamma} 2^{-k(\alpha-\gamma)} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma \\ &\lesssim \|\chi_E\|_{L^p}^2 2^{-(k-n)(\alpha-\gamma)} \left( \sup_i \frac{|E \cap I_i^k|}{|E| + |I_i^k|} \right)^\gamma. \end{aligned}$$

Thus if we choose  $\gamma > 0$  smaller than  $\min(1 - \alpha, \alpha)$  we obtain the desired conclusion after adding over  $k$

$$B(\chi_E, \chi_E) \lesssim \|\chi_E\|_{L^p}^2 \left( \sup_I \frac{|E \cap I|}{|E| + |I|} \right)^\gamma. \quad \square$$

By combining Lemma 4.4 and (4.3) we obtain that for  $f \in L^p$

$$B(f, f) \lesssim \|f\|_{L^p}^{2-\delta} \sup_{k,I} \|f_k\|_{L^p}^\delta \left( \frac{|E_k \cap I|}{|E_k| + |I|} \right)^{\delta\gamma/2},$$

that implies (after we rename  $\delta\gamma/2$  by  $\delta$ )

$$B(f, f) \lesssim \|f\|_{L^p}^{2-\delta} \sup_{k,I} \|f_k\|_{L^p}^\delta \left( \frac{|E_k \cap I|}{|E_k| + |I|} \right)^\delta,$$

since  $\|f_k\|_p / \|f\|_p \leq 1$  and so  $(\|f_k\|_p / \|f\|_p)^\delta \leq (\|f_k\|_p / \|f\|_p)^{\delta\gamma/2}$ .  $\square$

We note that  $\sup_{k,I} \|f_k\|_{L^p}^\delta \left( \frac{|E_k \cap I|}{|E_k| + |I|} \right)^\delta$  is bounded by  $(\sup_I |I|^{-1+1/p} \int_I |f|)^\delta$ . Indeed, we have

$$\sup_{k,I} \|f_k\|_{L^p}^\delta \left( \frac{|E_k \cap I|}{|E_k| + |I|} \right)^\delta \lesssim \left( \sup_{k,I} |I|^{-1+1/p} \int_I |f_k| \right)^\delta.$$

To see this, we rewrite  $\|f_k\|_{L^p}^\delta \asymp 2^{k\delta} |E_k|^{\delta/p}$  and  $\int_I |f_k| \asymp 2^k |E_k \cap I|$ . It suffices to show that for all  $k, I$

$$|E_k|^{\delta/p} \left( \frac{|E_k \cap I|}{|E_k| + |I|} \right)^\delta \leq |I|^{(-1+1/p)\delta} |E_k \cap I|^\delta,$$

which is equivalent to  $|E_k|^{\delta/p} |I|^\delta \leq |I|^{\delta/p} (|E_k| + |I|)^\delta$ . This holds trivially in the case  $|E_k| \leq |I|$ , while in the case  $|E_k| > |I|$  we rewrite the inequality as

$$1 \leq \left( 1 + \frac{|I|}{|E_k|} \right)^\delta \left( \frac{|I|}{|E_k|} \right)^{\delta(-1+1/p)},$$

which holds because  $-1 + 1/p < 0$ .

We have proved the following proposition.

**Proposition 4.5.** *Let  $0 < \alpha < 1$  and  $p = 2/(2 - \alpha)$ . There exist  $C < \infty$  and  $\delta \in (0, 2)$  such that for all  $f \in L^p(\mathbb{R})$*

$$B(f, f) \leq C \|f\|_{L^p(\mathbb{R})}^{2-\delta} \left( \sup_I |I|^{-1+1/p} \int_I |f| dx \right)^\delta. \quad (4.5)$$

where  $I$  ranges over all compact intervals of  $\mathbb{R}$ .

Using the Cauchy-Schwarz inequality for  $B$  and a decomposition as in Lemma 3.2 we obtain the corollary,

**Corollary 4.6.** *Let  $0 < \alpha < 1$  and  $p = 2/(2 - \alpha)$ . There exist  $C < \infty$  and  $\delta \in (0, 2)$  such that for all  $f \in L^p([0, 2\pi])$ ,*

$$\int_{[0, 2\pi]^2} f(x) f(y) |\sin(x - y)|^{-\alpha} dx dy \leq C \|f\|_{L^p([0, 2\pi])}^{2-\delta} \left( \sup_I |I|^{-1+1/p} \int_I |f| dx \right)^\delta,$$

where  $I$  ranges over all intervals of  $[0, 2\pi]$ .

We now consider the operator we will use to control the adjoint Fourier operator  $T$ .

**Definition 4.7.** Let  $0 < \alpha < 1$  and  $p = 2/(2 - \alpha)$ . We define the bilinear operator  $Q : L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2) \rightarrow \mathbb{R}$  by

$$Q(f, g) = \int_{(\mathbb{R}^2)^2} f(r, x)g(r', x')|x - x'|^{-\alpha} dx dx' dr dr', \quad (4.6)$$

Note that we can write  $Q(f, f) = B(\int f(r, x)dr, \int f(r', x')dr')$ .

For  $f \in L^p(r, x)$  with  $\|\int f(r, x)dr\|_{L_x^p} < \infty$  we use (4.5) to obtain

$$Q(f, f) \lesssim \left\| \int |f(r, x)|dr \right\|_{L_x^p}^{2-\delta} \left( \sup_I |I|^{-1+1/p} \int_I \int |f(r, x)|dr dx \right)^\delta. \quad (4.7)$$

Suppose that  $f(r, x)$  is supported where  $2^{k-1} \leq r < 2^k$ , then  $\int_I \int f(r, x)dr dx = \int_{\mathcal{C}} f(r, x)dr dx$ , where  $\mathcal{C} = [2^{k-1}, 2^k] \times I$ , and  $\|\int f(r, x)dr\|_{L_x^p} \leq 2^{(k-1)(1-1/p)} \|f\|_{L_{r,x}^p}$ . Thus, it follows from (4.7) that

$$Q(f, f) \lesssim 2^{2k(1-1/p)} \|f\|_{L_{r,x}^p}^{2-\delta} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1+1/p} \int_{\mathcal{C}} |f(r, x)|dr dx \right)^\delta, \quad (4.8)$$

where we used  $2^{k-1}|I| = |\mathcal{C}|$ .

In the case we are interested in we will need to estimate  $Q(|f_k|^{3/2}, |f_k|^{3/2})$  with the support of  $f_k$  as before and  $f_k \in L_{r,x}^2$ , with  $\alpha = 1/2$  and  $p = 4/3$ .

**Corollary 4.8.** *There exist  $C < \infty$  and  $\delta \in (0, 2)$  with the following property. Let  $k, k' \in \mathbb{Z}$  and  $f, g \in L^{4/3}(\mathbb{R}^2)$  and suppose that  $f(r, x), g(r, x)$  are supported in the regions  $[2^{k-1}, 2^k] \times \mathbb{R}$  and  $[2^{k'-1}, 2^{k'}] \times \mathbb{R}$  respectively. Then*

$$|Q(f, g)|^2 \leq C 2^{\frac{1}{2}(k+k')} \|f\|_{L_{r,x}^{4/3}}^{2-\delta} \|g\|_{L_{r,x}^{4/3}}^{2-\delta} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-\frac{1}{4}} \int_{\mathcal{C}} |f|dr dx \right)^\delta \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-\frac{1}{4}} \int_{\mathcal{C}} |g|dr dx \right)^\delta. \quad (4.9)$$

*Proof.* This follows from (4.8) and the Cauchy-Schwarz inequality for  $Q$ ,

$$Q(f, g)^2 \leq Q(f, f)Q(g, g). \quad \square$$

For  $f_k, f_{k'} \in L^2(\mathbb{R}_{(r,x)}^2)$  supported where  $2^{k-1} \leq r < 2^k$  and  $2^{k'-1} \leq r < 2^{k'}$  respectively we obtain

$$\begin{aligned} Q(|f_k|^{3/2}, |f_{k'}|^{3/2})^2 &\lesssim 2^{\frac{1}{2}(k+k')} \|f_k\|_{L_{r,x}^2}^{3(2-\delta)/2} \|f_{k'}\|_{L_{r,x}^2}^{3(2-\delta)/2} \\ &\quad \cdot \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2}dr dx \right)^\delta \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_{k'}|^{3/2}dr dx \right)^\delta. \end{aligned} \quad (4.10)$$

The use of the Cauchy-Schwarz inequality for  $Q$ , and a decomposition as in Lemma 3.2 implies that for  $f_k, f_{k'} \in L^2(\mathbb{R}_r \times [0, 2\pi]_x)$  supported where  $2^{k-1} \leq r < 2^k$  and

$2^{k'-1} \leq r < 2^{k'}$  the following estimate holds

$$\begin{aligned} \left( \int |f_k(r, x) f_{k'}(r', x')|^{3/2} |\sin(x - x')|^{-1/2} dx dx' dr dr' \right)^2 &\lesssim 2^{\frac{1}{2}(k+k')} \|f_k\|_{L^2_{r,x}}^{3(2-\delta)/2} \\ &\cdot \|f_{k'}\|_{L^2_{r,x}}^{3(2-\delta)/2} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} dr dx \right)^{\delta} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_{k'}|^{3/2} dr dx \right)^{\delta}. \end{aligned} \quad (4.11)$$

## 5. THE CAP BOUND FOR THE ADJOINT FOURIER RESTRICTION OPERATOR

**Proposition 5.1.** *There exist  $C < \infty$  and  $\delta \in (0, 2)$  with the following property. Let  $k, k' \in \mathbb{Z}$  and  $f, g \in L^2(\Gamma^2)$ , with  $f$  and  $g$  supported in the regions  $2^{k-1} \leq |y| < 2^k$  and  $2^{k'-1} \leq |y| < 2^{k'}$  respectively, then*

$$\begin{aligned} \|Tf \cdot Tg\|_{L^3(\mathbb{R}^3)} &\leq C 2^{-\frac{1}{6}|k-k'|} \|f\|_{L^2(\Gamma^2)}^{\frac{1}{2}(2-\delta)} \|g\|_{L^2(\Gamma^2)}^{\frac{1}{2}(2-\delta)} \\ &\cdot \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{\frac{3}{2}} d\sigma \right)^{\frac{\delta}{3}} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |g|^{\frac{3}{2}} d\sigma \right)^{\frac{\delta}{3}}. \end{aligned} \quad (5.1)$$

*Proof.* Recall from Section 3, equation (3.3), that we have the inequality

$$\begin{aligned} \|Tf \cdot Tg\|_{L^3} &\leq C (2^k 2^{k'})^{-1/3} \min(2^k, 2^{k'})^{1/3} \\ &\cdot \left( \int |f(y)g(y')|^{3/2} |\sin(\theta - \theta')|^{-1/2} d\theta d\theta' dr dr' \right)^{2/3}. \end{aligned}$$

From (4.11) we obtain

$$\begin{aligned} \|Tf \cdot Tg\|_{L^3} &\lesssim (2^k 2^{k'})^{-1/3} \min(2^k, 2^{k'})^{1/3} 2^{(k+k')/6} \|f\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \|g\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \\ &\cdot \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} \right)^{\delta/3} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |g|^{3/2} \right)^{\delta/3}, \end{aligned}$$

which as in the proof of Lemma 3.3 can be rewritten as

$$\begin{aligned} \|Tf \cdot Tg\|_{L^3} &\lesssim 2^{-|k-k'|/6} \|f\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \|g\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \\ &\cdot \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} \right)^{\delta/3} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |g|^{3/2} \right)^{\delta/3}. \quad \square \end{aligned}$$

**Corollary 5.2.** *There exist  $C < \infty$  and  $\delta \in (0, 2)$  with the following property. If  $f \in L^2(\Gamma^2)$  and  $f_k = f \chi_{\{2^{k-1} \leq |y| < 2^k\}}$ ,  $k \in \mathbb{Z}$ , then*

$$\|Tf\|_{L^6(\mathbb{R}^3)}^2 \leq C \sum_{k \in \mathbb{Z}} \|f_k\|_{L^2(\Gamma^2)}^{2-\delta} \left( \sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} d\sigma \right)^{2\delta/3}. \quad (5.2)$$

*Proof.* We start by writing  $\|Tf\|_{L^6}^2 = \|Tf \cdot Tf\|_{L^3}$  and  $Tf = \sum_{k \in \mathbb{Z}} Tf_k$ , so the triangle inequality gives

$$\|Tf \cdot Tf\|_{L^3} \leq \sum_{k, k'} \|Tf_k \cdot Tf_{k'}\|_{L^3}$$

that together with Proposition 5.1 gives

$$\begin{aligned} \|Tf\|_{L^6}^2 &\lesssim \sum_{k,k'} 2^{-|k-k'|/6} \|f_k\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \|f_{k'}\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \\ &\quad \cdot \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f_k|^{3/2} \right)^{\delta/3} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f_{k'}|^{3/2} \right)^{\delta/3}. \end{aligned}$$

The desired conclusion follows by the Cauchy-Schwarz inequality.  $\square$

By using Proposition 5.1 instead of Lemma 3.3 we can obtain an analog of Proposition 3.4, that is

**Proposition 5.3.** *There exist  $C < \infty$  and  $\delta \in (0, 2)$  with the following property. Let  $f \in L^2(\Gamma^2)$  and for  $k \in \mathbb{Z}$  let  $f_k(y) = f(y)\chi_{\{2^{k-1} \leq |y| < 2^k\}}$ . Then*

$$\|Tf\|_{L^6(\mathbb{R}^3)} \leq C \left( \sum_{k \in \mathbb{Z}} \|f_k\|_{L^2(\Gamma^2)}^{3-3\delta/2} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f_k|^{3/2} d\sigma \right)^\delta \right)^{1/3}. \quad (5.3)$$

**Proposition 5.4** (Cap estimate). *There exist  $C < \infty$  and  $\delta \in (0, 2)$  such that for all  $f \in L^2(\Gamma^2)$  the following estimate holds*

$$\|Tf\|_{L^6(\mathbb{R}^3)} \leq C \|f\|_{L^2(\Gamma^2)}^{1-\delta/2} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f|^{3/2} d\sigma \right)^{\delta/3}, \quad (5.4)$$

*Proof.* From Proposition 5.3 we have

$$\|Tf\|_{L^6} \lesssim \left( \sum_k \|f_k\|_{L^2}^{3-3\delta/2} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f_k|^{3/2} d\sigma \right)^\delta \right)^{1/3}.$$

For each  $k$ , using that  $\delta \leq 2/3$  ( $\delta$  can be taken as small as desired by changing the corresponding implicit constants  $C$  in the inequalities) we have

$$\begin{aligned} \|f_k\|_{L^2}^{3-3\delta/2} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f_k|^{3/2} d\sigma \right)^\delta &= \|f_k\|_{L^2}^2 \|f_k\|_{L^2}^{1-3\delta/2} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f_k|^{3/2} d\sigma \right)^\delta \\ &\leq \|f_k\|_{L^2}^2 \|f\|_{L^2}^{1-3\delta/2} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f|^{3/2} d\sigma \right)^\delta. \end{aligned}$$

Then, adding over  $k$ ,

$$\|Tf\|_{L^6} \lesssim \|f\|_{L^2}^{1-\delta/2} \left( \sup_c |\mathcal{C}|^{-1/4} \int_c |f|^{3/2} d\sigma \right)^{\delta/3}. \quad \square$$

## 6. USING THE CAP BOUND

We will prove the analog of [2, Lemma 2.6].

**Lemma 6.1.** *For any  $\delta > 0$  there exist  $C_\delta < \infty$  and  $\eta_\delta > 0$  with the following property. If  $f \in L^2(\Gamma^2)$  satisfies  $\|Tf\|_6 \geq \delta C \|f\|_2$  then there exists a decomposition*

$f = g + h$  and a cap  $\mathcal{C}$  satisfying

$$0 \leq |g|, |h| \leq |f|, \quad (6.1)$$

$$g, h \text{ have disjoint supports}, \quad (6.2)$$

$$|g(x)| \leq C_\delta \|f\|_2 |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x), \text{ for all } x, \quad (6.3)$$

$$\|g\|_2 \geq \eta_\delta \|f\|_2. \quad (6.4)$$

*Proof.* For convenience, normalize so that  $\|f\|_{L^2(\Gamma^2)} = 1$ . By Proposition 5.4 there exists a cap  $\mathcal{C}$  such that

$$\int_{\mathcal{C}} |f|^{3/2} dr d\theta \geq \frac{1}{2} c(\delta) |\mathcal{C}|^{1/4}.$$

Let  $R \geq 1$  and define  $E = \{x \in \mathcal{C} : |f(x)| \leq R\}$ . Set  $g = f\chi_E$  and  $h = f - f\chi_E$ . Then  $g, h$  have disjoint supports,  $g + h = f$ ,  $g$  is supported on  $\mathcal{C}$ , and  $\|g\|_\infty \leq R$ . Since  $|h(x)| \geq R$  for almost every  $x \in \mathcal{C}$  for which  $h(x) \neq 0$  we have

$$\int_{\mathcal{C}} |h|^{3/2} \leq R^{-1/2} \int_{\mathcal{C}} h^2 \leq R^{-1/2} \|f\|_2^2 = R^{-1/2}.$$

If we choose  $R$  by setting  $R^{-1/2} = \frac{1}{4} c(\delta) |\mathcal{C}|^{1/4}$ , then

$$\int_{\mathcal{C}} |g|^{3/2} = \int_{\mathcal{C}} |f|^{3/2} - \int_{\mathcal{C}} |h|^{3/2} \geq \frac{1}{4} c(\delta) |\mathcal{C}|^{1/4}.$$

By Hölder's inequality, since  $g$  is supported on  $\mathcal{C}$ ,

$$\|g\|_2 \geq |\mathcal{C}|^{-1/6} \left( \int_{\mathcal{C}} |g|^{3/2} \right)^{2/3} \geq c'(\delta) = c'(\delta) \|f\|_2 > 0. \quad \square$$

We note that the conditions  $|g(x)| \leq C_\delta \|f\|_2 |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x)$  and  $\|g\|_2 \geq \eta_\delta \|f\|_2$  easily imply a lower bound on the  $L^1$  norm of  $g$ .

**Lemma 6.2.** *Let  $g \in L^2(\Gamma^2)$  satisfying  $|g(x)| \leq a |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x)$  and  $\|g\|_2 \geq b$ , for some  $a, b > 0$  and  $\mathcal{C} \subset \Gamma^2$ . Then there is a constant  $C = C(a, b) > 0$  such that*

$$\|g\|_{L^1(\Gamma^2)} \geq C |\mathcal{C}|^{1/2}.$$

*Proof.* The hypotheses on  $g$  imply that  $|a^{-1} |\mathcal{C}|^{1/2} g(x)| \leq \chi_{\mathcal{C}}(x) \leq 1$  and thus  $\|a^{-1} |\mathcal{C}|^{1/2} g\|_2^2 \leq \|a^{-1} |\mathcal{C}|^{1/2} g\|_1$ . Therefore

$$\|g\|_1 \geq a^{-1} |\mathcal{C}|^{1/2} \|g\|_2^2 \geq a^{-1} b^2 |\mathcal{C}|^{1/2}. \quad \square$$

## 7. USING THE GROUP OF SYMMETRIES

A Lorentz transformation,  $L$ , in  $\mathbb{R}^3$  is an invertible linear map that preserves the bilinear form

$$A(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

$x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ , i.e.

$$A(x, y) = A(Lx, Ly), \text{ for all } x, y \in \mathbb{R}^3.$$



Examples of Lorentz transformations are  $L^t$ ,  $M^t$  and  $R_\theta$  given next. For  $t \in (-1, 1)$  we define the linear map  $L^t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$L^t(x_1, x_2, x_3) = \left( \frac{x_1 + tx_3}{\sqrt{1-t^2}}, x_2, \frac{x_3 + tx_1}{\sqrt{1-t^2}} \right).$$

$\{L^t\}_{t \in (-1, 1)}$  is a one parameter subgroup of Lorentz transformations. Similarly,

$$M^t(x_1, x_2, x_3) = \left( x_1, \frac{x_2 + tx_3}{\sqrt{1-t^2}}, \frac{x_3 + tx_2}{\sqrt{1-t^2}} \right)$$

is a Lorentz transformation.

One computes that  $L^t$  and  $M^t$  preserve the cone for all  $t \in (-1, 1)$ , that is,  $L^t(\Gamma^2) = M^t(\Gamma^2) = \Gamma^2$ . For  $\lambda > 0$  we define the dilation  $D_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by  $D_\lambda(x) = \lambda x$  that clearly satisfies  $D_\lambda(\Gamma^2) = \Gamma^2$  for every  $\lambda > 0$ . For  $\theta \in [0, 2\pi]$  we denote by  $R_\theta$  the rotation in  $\mathbb{R}^3$  by angle  $\theta$  about the  $x_3$ -axis

$$R_\theta(x_1, x_2, x_3) = (x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta, x_3).$$

$R_\theta$  preserves the cone for all  $0 \leq \theta \leq 2\pi$ .

Associated to  $L^t$ ,  $M^t$ ,  $D_\lambda$ ,  $R_\theta$  are the operators  $L^{t*}$ ,  $M^{t*}$ ,  $D_\lambda^*$  and  $R_\theta^*$  acting on a function  $f \in L^2(\Gamma^2)$  by

$$L^{t*}f = f \circ L^t, \quad M^{t*}f = f \circ M^t, \quad D_\lambda^*f = \lambda^{1/2}f \circ D_\lambda, \quad R_\theta^*f = f \circ R_\theta, \quad (7.1)$$

where “ $\circ$ ” denotes composition. We also define  $L_t = \sqrt{1-t^2}L^t = D_{\sqrt{1-t^2}}L^t$  and  $L_t^*$  by

$$L_t^*f(x_1, x_2, x_3) = (1-t^2)^{\frac{1}{4}}f \circ L^t(x_1, x_2, x_3) = (1-t^2)^{\frac{1}{4}}f(x_1 + tx_3, \sqrt{1-t^2}x_2, x_3 + tx_1).$$

The measure  $\sigma$  is invariant under the action of Lorentz transformations that preserve the cone, and in fact is the only one with that property, up to multiplication by constant; for this we refer to [9] where the case of the cone in  $\mathbb{R}^4$  is considered. In this paper we only need to know that for every  $t \in (-1, 1)$ ,  $L^t$  and  $M^t$  preserve the measure  $\sigma$  and this can be done directly using the change of variables formula and seeing that the Jacobians work out. We write it in the next proposition and include the proof for completeness.

**Proposition 7.1.** *For any  $t \in (-1, 1)$  the linear maps  $L^t$ ,  $M^t$  are invertible, preserve  $\Gamma^2$ , that is  $L^t(\Gamma^2) = M^t(\Gamma^2) = \Gamma^2$ , and preserve  $\sigma$ , that is, for any  $f \in L^1(\Gamma^2)$*

$$\int_{\Gamma^2} f \circ L^t d\sigma = \int_{\Gamma^2} f \circ M^t d\sigma = \int_{\Gamma^2} f d\sigma.$$

*Proof.* Letting  $P(x_1, x_2, x_3) = (x_2, x_1, x_3)$  we see that  $M^t = P \circ L^t \circ P$  and so it is enough to prove the statements for  $L^t$ . The inverse of  $L^t$  is  $L^{-t}$ . That  $L^t(\Gamma^2) \subseteq \Gamma^2$  follows from the equality

$$\left( \frac{x_3 + tx_1}{\sqrt{1-t^2}} \right)^2 = \left( \frac{x_1 + tx_3}{\sqrt{1-t^2}} \right)^2 + x_2^2$$

and the inequality

$$\frac{x_3 + tx_1}{\sqrt{1-t^2}} \geq 0$$

whenever  $x_3^2 = x_1^2 + x_2^2$  and  $x_3 \geq 0$ . Since the same is true for  $L^{-t}$ , it follows that  $L^t(\Gamma^2) = \Gamma^2$ . For the invariance of the measure, let  $f \in L^1(\Gamma^2)$ . We have

$$\int f \circ L^t(x_1, x_2, x_3) d\sigma(x_1, x_2, x_3) = \int_{\mathbb{R}^2} f\left(\frac{y_1 + ty_3}{\sqrt{1-t^2}}, y_2, \frac{y_3 + ty_1}{\sqrt{1-t^2}}\right) \frac{dy_1 dy_2}{\sqrt{y_1^2 + y_2^2}}$$

where  $y_3 = \sqrt{y_1^2 + y_2^2}$ . We use the change of variables  $u = \frac{y_1 + ty_3}{\sqrt{1-t^2}} = \frac{y_1 + t\sqrt{y_1^2 + y_2^2}}{\sqrt{1-t^2}}$ ,  $v = y_2$ . We note that the Jacobian is

$$\frac{\partial(u, v)}{\partial(y_1, y_2)} = \frac{1}{\sqrt{1-t^2}} \left(1 + \frac{ty_1}{\sqrt{y_1^2 + y_2^2}}\right),$$

or equivalently

$$\frac{\partial(y_1, y_2)}{\partial(u, v)} = \sqrt{y_1^2 + y_2^2} \frac{\sqrt{1-t^2}}{\sqrt{y_1^2 + y_2^2} + ty_1}.$$

Now, since  $L^t(y_1, y_2, y_3)$  lies in  $\Gamma^2$  we also have

$$\sqrt{u^2 + v^2} = \frac{y_3 + ty_1}{\sqrt{1-t^2}}.$$

It follows that the Jacobian factor can be rewritten as

$$\frac{\partial(y_1, y_2)}{\partial(u, v)} = \frac{\sqrt{y_1^2 + y_2^2}}{\sqrt{u^2 + v^2}}.$$

Therefore, letting  $w = \sqrt{u^2 + v^2}$ ,

$$\int_{\mathbb{R}^2} f\left(\frac{y_1 + ty_3}{\sqrt{1-t^2}}, y_2, \frac{y_3 + ty_1}{\sqrt{1-t^2}}\right) \frac{dy_1 dy_2}{\sqrt{y_1^2 + y_2^2}} = \int_{\mathbb{R}^2} f(u, v, w) \frac{du dv}{\sqrt{u^2 + v^2}}$$

or equivalently,

$$\int f \circ L^t d\sigma = \int f d\sigma. \quad \square$$

The Lorentz invariance of the measure implies invariance of the  $L^2$  norm, for  $f \in L^2(\Gamma^2)$

$$\|L^{t*} f\|_{L^2(\sigma)} = \|M^{t*} f\|_{L^2(\sigma)} = \|D_\lambda^* f\|_{L^2(\sigma)} = \|R_\theta^* f\|_{L^2(\sigma)} = \|L_t^* f\|_{L^2(\sigma)} = \|f\|_{L^2(\sigma)}. \quad (7.2)$$

Using the Lorentz invariance of  $\sigma$  it is direct to check that for all  $p \in [1, \infty]$  the  $L^p$  norm of  $Tf$  does not change under Lorentz transformations in the sense that

$$\|T(f \circ L)\|_{L^p(\mathbb{R}^3)} = \|Tf\|_{L^p(\mathbb{R}^3)}. \quad (7.3)$$

Indeed, writing

$$Tf(x, t) = \int e^{ix \cdot y} e^{ity'} f(y, y') d\sigma(y, y') = \int e^{iA((x, -t), (y, y'))} f(y, y') d\sigma(y, y'),$$

thus

$$\begin{aligned} T(f \circ L)(x, t) &= \int e^{iA((x, -t), (y, y'))} f \circ L(y, y') d\sigma(y, y') \\ &= \int e^{iA(L(x, -t), L(y, y'))} f \circ L(y, y') d\sigma(y, y') \\ &= \int e^{iA(L(x, -t), (y, y'))} f(y, y') d\sigma(y, y'). \end{aligned}$$

Since for a Lorentz transformation  $L$ ,  $|\det L| = 1$ , (7.3) follows by change of variables in the case  $p \in [1, \infty)$ . When  $p = \infty$ , (7.3) follows since  $L$  is invertible.

We can use the group of symmetries to widen caps, that is, we have

**Lemma 7.2.** *Let  $\mathcal{C} \subset [1/2, 1] \times [0, 2\pi]$  be a cap in  $\Gamma^2$ . Then there exist  $t \in [0, 1]$  and  $\theta \in [0, 2\pi]$  such that  $L_t^{-1}R_\theta^{-1}(\mathcal{C})$  satisfies*

$$\sigma(L_t^{-1}R_\theta^{-1}(\mathcal{C})) \geq \frac{1}{2}, \text{ and } L_t^{-1}R_\theta^{-1}(\mathcal{C}) \subseteq [1/4, 1] \times [0, 2\pi]. \quad (7.4)$$

*Proof.* Let  $\theta \in [0, 2\pi]$  be such that  $R_\theta^{-1}\mathcal{C} = [1/2, 1] \times [-\varepsilon, \varepsilon]$ , for some  $\varepsilon \in [0, \pi]$ . The measure of  $\mathcal{C}$  is  $\sigma(\mathcal{C}) = |\mathcal{C}| = \varepsilon$ , and so we can assume  $\varepsilon < 1/2$ , otherwise we are done by taking  $t = 0$ .

The inverse of  $L_t$  is  $L_t^{-1} = (1 - t^2)^{-1/2}L^{-t}$  and the measure of  $L_t^{-1}R_\theta^{-1}(\mathcal{C})$  is

$$\begin{aligned} \sigma(L_t^{-1}R_\theta^{-1}(\mathcal{C})) &= \sigma((1 - t^2)^{-1/2}L^{-t}R_\theta^{-1}(\mathcal{C})) = \sigma((L^t)^{-1}R_\theta^{-1}((1 - t^2)^{-1/2}\mathcal{C})) \\ &= \sigma((1 - t^2)^{-1/2}\mathcal{C}) = (1 - t^2)^{-1/2}\sigma(\mathcal{C}), \end{aligned}$$

where we used the invariance of  $\sigma$  under Lorentz transformations and that  $\sigma(\lambda\mathcal{C}) = |\lambda|\sigma(\mathcal{C})$  for any  $\lambda \in \mathbb{R}$ .

Let  $t$  be such that  $\sigma(L_t^{-1}R_\theta^{-1}(\mathcal{C})) = 1$ , that is  $t = (1 - |\mathcal{C}|^2)^{1/2} = (1 - \varepsilon^2)^{1/2}$ . Now we write  $R_\theta^{-1}\mathcal{C} = \{(r \cos \varphi, r \sin \varphi, r) : 1/2 \leq r \leq 1, -\varepsilon \leq \varphi \leq \varepsilon\}$ , so that

$$L_t^{-1}R_\theta^{-1}(\mathcal{C}) = \left\{ r(1 - t^2)^{-1/2} \left( \frac{\cos \varphi - t}{(1 - t^2)^{1/2}}, \sin \varphi, \frac{1 - t \cos \varphi}{(1 - t^2)^{1/2}} \right) : \frac{1}{2} \leq r \leq 1, -\varepsilon \leq \varphi \leq \varepsilon \right\}.$$

Note that for  $1/2 \leq r \leq 1$  and  $-\varepsilon \leq \varphi \leq \varepsilon$  we have

$$r(1 - t^2)^{-1/2} \frac{|\cos \varphi - t|}{(1 - t^2)^{1/2}} \leq \frac{1 - t}{1 - t^2} = \frac{1}{1 + t} \leq 1$$

because  $\cos \varphi \geq \cos \varepsilon \geq t$ . Similarly

$$r(1 - t^2)^{-1/2} |\sin \varphi| \leq \frac{\sin \varepsilon}{\varepsilon} \leq 1$$

and

$$\frac{1}{4} \leq \frac{1}{2(1 + t)} = \frac{1 - t}{2(1 - t^2)} \leq r \frac{1 - t \cos \varphi}{1 - t^2} \leq 1.$$

Then  $t = (1 - \varepsilon^2)^{1/2}$  gives the desired conclusion.  $\square$

**Corollary 7.3.** *Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of nonnegative functions in  $L^2(\Gamma^2)$  with  $\|f_n\|_{L^2(\Gamma^2)} = 1$  and such that there exists a cap  $\mathcal{C}_n \subset [1/2, 1] \times [0, 2\pi]$  with the property*

$$\int_{\mathcal{C}_n} f_n d\sigma \geq c |\mathcal{C}_n|^{1/2}, \quad (7.5)$$

*where  $c > 0$  is independent of  $n$ . Then there exist sequences  $\{t_n\}_{n \in \mathbb{N}} \subset [0, 1)$  and  $\{\theta_n\}_{n \in \mathbb{N}} \subset [0, 2\pi]$  such that  $\{L_{t_n}^* R_{\theta_n}^* f_n\}_{n \in \mathbb{N}}$  satisfies that every weak limit in  $L^2(\Gamma^2)$  is nonzero.*

*Proof.*  $L_t^*$  and  $R_\theta^*$  preserve the  $L^2(\Gamma^2)$  norm thus  $\|L_t^* R_\theta^* f_n\|_{L^2(\Gamma^2)} = 1$  for any  $t \in [0, 1)$  and  $\theta \in [0, 2\pi]$ . It follows that for any sequences  $\{t_n\}_{n \in \mathbb{N}} \subset [0, 1)$  and  $\{\theta_n\}_{n \in \mathbb{N}} \subset [0, 2\pi]$ , the set of  $L^2$ -weak limits of  $\{L_{t_n}^* R_{\theta_n}^* f_n\}$  is nonempty.

Under the action of  $L_t^* R_\theta^*$  the integral of a function  $f$  changes according to

$$\int L_t^* R_\theta^* f d\sigma = (1 - t^2)^{-1/4} \int R_\theta^* f d\sigma = (1 - t^2)^{-1/4} \int f d\sigma. \quad (7.6)$$

By Lemma 7.2, for each  $n$  there exist  $t_n \in [0, 1)$  and  $\theta_n \in [0, 2\pi]$  such that

$$\sigma(L_{t_n}^{-1} R_{\theta_n}^{-1}(\mathcal{C}_n)) \geq \frac{1}{2} \text{ and } L_{t_n}^{-1} R_{\theta_n}^{-1}(\mathcal{C}_n) \subseteq [1/4, 1] \times [0, 2\pi].$$

Suppose that for a subsequence (that we call the same)  $L_{t_n}^* R_{\theta_n}^* f_n \rightharpoonup f$ , as  $n \rightarrow \infty$ , for some  $f \in L^2(\Gamma^2)$ . Using (7.5) and (7.6) we have

$$\begin{aligned} \int_{[1/4, 1] \times [0, 2\pi]} L_{t_n}^* R_{\theta_n}^* f_n d\sigma &\geq (1 - t_n^2)^{-1/4} \int_{\mathcal{C}_n} f_n d\sigma \\ &\geq c(1 - t_n^2)^{-1/4} |\mathcal{C}_n|^{1/2} = c(\sigma(L_{t_n}^{-1} R_{\theta_n}^{-1}(\mathcal{C}_n)))^{1/2} \geq \frac{c}{\sqrt{2}}. \end{aligned} \quad (7.7)$$

From (7.7) and the weak convergence it follows that

$$\int_{[1/4, 2] \times [0, 2\pi]} f d\sigma \geq \frac{c}{\sqrt{2}} > 0$$

and so  $f \neq 0$ . □

## 8. THE PROOF OF THE PRECOMPACTNESS

In this section we prove that up to symmetries of the cone, an extremizing sequence is precompact.

We will give two proofs. In this section the proof will be based on [4] and [5]; the other will be based on [2] with a modification coming from [4] and is given in Section 9.

Recall that  $\mathbf{C}$ , given in (1.3), denotes the best constant in the inequality (1.2), in other words,  $\mathbf{C} = \|T\|$ , the norm of the operator  $T$  as a map from  $L^2(\Gamma^2)$  to  $L^6(\mathbb{R}^3)$ .

We start by stating Proposition 1.1 of [4] for the cone.

**Proposition 8.1** ([4]). *Let  $T : L^2(\Gamma^2, \sigma) \rightarrow L^6(\mathbb{R}^3)$  be the Fourier extension operator defined in (1.1). Let  $\{f_n\}_{n \in \mathbb{N}} \subset L^2(\Gamma^2)$  such that:*

- (i)  $\lim_{n \rightarrow \infty} \|f_n\|_2 = 1$ ;

- (ii)  $\lim_{n \rightarrow \infty} \|Tf_n\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$ ;
- (iii)  $f_n \rightharpoonup f \neq 0$ ;
- (iv)  $Tf_n \rightarrow Tf$  a.e. in  $\mathbb{R}^3$ .

Then  $f_n \rightarrow f$  in  $L^2(\Gamma^2)$ , in particular  $\|f\|_2 = 1$  and  $\|Tf\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$ .

We have changed slightly condition (i) in Proposition 1.1 of [4] from  $\|f_n\|_2 = 1$  to  $\lim_{n \rightarrow \infty} \|f_n\|_2 = 1$ , but the proposition as stated here is easily shown to be equivalent to the one in [4] by considering  $f_n/\|f_n\|_2$ . Note that an extremizing sequence  $\{f_n\}_{n \in \mathbb{N}}$  as in Definition 1.2 satisfies (i) and (ii) in the previous proposition.

We now restate the precompactness theorem, Theorem 1.3, in a more precise way,

**Theorem 8.2.** *Let  $\{f_n\}_{n \in \mathbb{N}}$  be an extremizing sequence for (1.2) of nonnegative functions in  $L^2(\Gamma^2)$ . Then there exist sequences  $\{t_n\}_{n \in \mathbb{N}} \subset (-1, 1)$ ,  $\{\theta_n\}_{n \in \mathbb{N}} \subset [0, 2\pi]$  and  $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \infty)$  such that  $\{L_{t_n}^* R_{\theta_n}^* D_{\lambda_n}^* f_n\}_{n \in \mathbb{N}}$  is precompact, that is, any subsequence has a convergent sub-subsequence in  $L^2(\Gamma^2)$ .*

*Proof.* Since  $\{f_n\}_{n \in \mathbb{N}}$  is an extremizing sequence, for all  $n$  large enough  $\|Tf_n\|_6 \geq \frac{C}{2} \|f_n\|_2$ . By Lemma 6.1 with  $\delta = 1/2$  there exists  $C < \infty$  and  $\eta > 0$ , a decomposition  $f_n = g_n + h_n$  and a cap  $\mathcal{C}_n$  satisfying (6.1), (6.2), (6.3) and (6.4). Using that  $\|f_n\|_{L^2} \rightarrow 1$ , as  $n \rightarrow \infty$ , and Lemma 6.2 for  $g_n$  gives

$$\|g_n\|_{L^1(\Gamma^2)} \geq C |\mathcal{C}_n|^{1/2},$$

where  $C$  is independent of  $n$ .

Now there exists  $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \infty)$  such that  $\lambda_n^{-1} \mathcal{C}_n \subset [1/2, 1] \times [0, 2\pi]$  and  $\lambda_n^{-1} \mathcal{C}_n$  is a cap as in Definition 4.1. By dilation invariance  $\{D_{\lambda_n}^* f_n\}_{n \in \mathbb{N}}$  is also an extremizing sequence, with  $\|D_{\lambda_n}^* f_n\|_2 = \|f_n\|_2$ . The decomposition for  $f_n$  gives a decomposition for  $D_{\lambda_n}^* f_n$ ,  $D_{\lambda_n}^* f_n = D_{\lambda_n}^* g_n + D_{\lambda_n}^* h_n$ , and

$$\int_{\lambda_n^{-1} \mathcal{C}_n} D_{\lambda_n}^* f_n d\sigma \geq \|D_{\lambda_n}^* g_n\|_{L^1(\Gamma^2)} \geq C |\lambda_n^{-1} \mathcal{C}_n|^{1/2}.$$

We now apply Corollary 7.3 to  $\{D_{\lambda_n}^* f_n\}_n$  to obtain sequences  $\{t_n\}_{n \in \mathbb{N}} \subset [0, 1]$  and  $\{\theta_n\}_{n \in \mathbb{N}} \subset [0, 2\pi]$  such that every weak limit of  $\{L_{t_n}^* R_{\theta_n}^* D_{\lambda_n}^* f_n\}_n$  in  $L^2(\Gamma^2)$  is nonzero.

In view of Proposition 8.1 the theorem is proved if we show that, after passing to a subsequence, if  $L_{t_n}^* R_{\theta_n}^* D_{\lambda_n}^* f_n \rightharpoonup f$ , as  $n \rightarrow \infty$ , then  $TL_{t_n}^* R_{\theta_n}^* D_{\lambda_n}^* f_n \rightarrow Tf$  a.e. in  $\mathbb{R}^3$ . We will do this by using the following proposition.  $\square$

**Proposition 8.3.** *Let  $\{u_n\}_{n \in \mathbb{N}}$  be a uniformly bounded sequence in  $L^2(\Gamma^2)$ , i.e.,  $\sup_n \|u_n\|_2 =: c < \infty$ . Suppose there exists  $u \in L^2(\Gamma^2)$  such that  $u_n \rightharpoonup u$  as  $n \rightarrow \infty$ . Then, there exists a subsequence  $\{u_{n_k}\}_{k \in \mathbb{N}}$  such that  $Tu_{n_k} \rightarrow Tu$  a.e. in  $\mathbb{R}^3$ .*

*Proof.* The proof of this is contained in the proof of Theorem 1.1 in [5]. We repeat it here for the convenience of the reader (and the author). We start by defining  $v_n(y)$  by its Fourier transform

$$\hat{v}_n(y) = u_n(y)|y|^{-1},$$

and  $\hat{v}(y) = u(y)|y|^{-1}$ .

Since  $\|u_n\|_{L^2(\Gamma^2)}^2 = \int_{\mathbb{R}^2} |u_n(y)|^2 \frac{dy}{|y|} \leq c^2$  we see that  $\|v_n\|_{\dot{H}^{1/2}(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |\hat{v}_n(y)|^2 |y| dy \leq c^2$ . The operator  $T$  applied to  $u_n$  equals  $(2\pi)^2 e^{it\sqrt{-\Delta}} v_n$ . Fix  $t \in \mathbb{R}$ , by the continuity of  $e^{it\sqrt{-\Delta}}$  in  $\dot{H}^{1/2}(\mathbb{R}^2)$ , we have

$$e^{it\sqrt{-\Delta}} v_n \rightharpoonup e^{it\sqrt{-\Delta}} v$$

weakly in  $\dot{H}^{1/2}(\mathbb{R}^2)$ , as  $n \rightarrow \infty$ . Then, by the Rellich Theorem ([3, Theorem 1.5]), for any  $R > 0$

$$e^{it\sqrt{-\Delta}} v_n \rightarrow e^{it\sqrt{-\Delta}} v$$

strongly in  $L^2(B(0, R))$ , as  $n \rightarrow \infty$ . Denote by

$$F_n(t) := \int_{|x| < R} \left| e^{it\sqrt{-\Delta}}(v_n - v) \right|^2 dx = \|e^{it\sqrt{-\Delta}}(v_n - v)\|_{L^2(B(0, R))}^2.$$

By Hölder's inequality and Sobolev embedding,  $\dot{H}^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ , we obtain

$$F_n(t) \leq CR \|e^{it\sqrt{-\Delta}}(v_n - v)\|_{\dot{H}^{1/2}(\mathbb{R}^2)}^2 \leq 2CR,$$

consequently, by the Fubini and dominated convergence Theorems we have that

$$\int_{-R}^R F_n(t) dt = \int_{-R}^R \int_{|x| < R} \left| e^{it\sqrt{-\Delta}}(v_n - v) \right|^2 dx dt \rightarrow 0$$

as  $n \rightarrow \infty$ . This implies that, up to a subsequence,

$$e^{it\sqrt{-\Delta}}(v_n - v) \rightarrow 0 \quad \text{a.e. in } B(0, R) \times (-R, R).$$

Repeating the argument on a discrete sequence of radii  $R_n$  such that  $R_n \rightarrow \infty$ , as  $n \rightarrow \infty$  we conclude, by a diagonal argument, that there exists a subsequence  $v_{n_k}$  of  $v_n$  such that

$$e^{it\sqrt{-\Delta}}(v_{n_k} - v)(x) \rightarrow 0 \quad \text{a.e. for } (x, t) \in \mathbb{R}^2 \times \mathbb{R},$$

or equivalently, in terms of the sequence  $\{u_n\}_{n \in \mathbb{N}}$ ,

$$Tu_{n_k} - Tv \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^3. \quad \square$$

This concludes the proof of Theorem 8.2. For the proof of Theorem 8.2 using the Christ-Shao argument we will need the next proposition, an analog of Proposition 8.1, which of course follows from Propositions 8.1 and 8.3, but the idea is to give an alternative approach.

We denote  $B(0, R)^c := \{x \in \mathbb{R}^2 : |x| \geq R\}$ , the complement of the ball  $B(0, R)$ .

**Proposition 8.4.** *Let  $T : L^2(\Gamma^2, \sigma) \rightarrow L^6(\mathbb{R}^3)$  be the Fourier extension operator defined in (1.1). Let  $\{f_n\}_{n \in \mathbb{N}} \subset L^2(\Gamma^2)$  such that:*

- (i)  $\lim_{n \rightarrow \infty} \|f_n\|_2 = 1$ ;
- (ii)  $\lim_{n \rightarrow \infty} \|Tf_n\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$ ;
- (iii)  $f_n \rightharpoonup f \neq 0$ ;
- (iv)  $\sup_{n \in \mathbb{N}} \|f_n\|_{L^2(B(0, R)^c)} \leq \Theta(R)$ , where  $\Theta(R) \rightarrow 0$  as  $R \rightarrow \infty$ .

*Then  $f_n \rightarrow f$  in  $L^2(\Gamma^2)$ , in particular  $\|f\|_2 = 1$  and  $\|Tf\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$ .*

*Proof.* Our proof follows that of Proposition 1.1 in [4]. We will denote by  $o_n(1)$  a quantity depending on  $n$  only that satisfies  $\lim_{n \rightarrow \infty} o_n(1) = 0$ . We will allow  $o_n(1)$  to change from line to line without changing its name.

Let  $R > 0$ . Note that because of the weak convergence we also have  $\|f\|_{L^2(B(0,R)^c)} \leq \Theta(R)$ . Denote  $f^R = f\chi_{B(0,R)}$  and  $f_n^R = f_n\chi_{B(0,R)}$ . Because of the weak convergence and the compact support of  $f^R$  and  $f_n^R$ , we have

$$T(f_n^R) \rightarrow T(f^R), \text{ a.e. in } \mathbb{R}^3,$$

and because of the continuity of  $T$ ,

$$\|T(f_n - f_n^R)\|_6, \|T(f - f^R)\|_6 \leq C\Theta(R). \quad (8.1)$$

Thus by triangular inequality, using that  $\|Tf_n^R - Tf^R\|_6 \leq C$  for all  $n$  and the binomial expansion

$$\begin{aligned} \|Tf_n - Tf\|_6^6 &\leq (\|Tf_n - Tf_n^R\|_6 + \|T(f_n^R - f^R)\|_6 + \|T(f^R - f)\|_6)^6 \\ &\leq \|Tf_n^R - Tf^R\|_6^6 + C\Theta(R), \end{aligned} \quad (8.2)$$

and similarly

$$\|Tf_n^R - Tf^R\|_6^6 \leq \|Tf_n - Tf\|_6^6 + C\Theta(R). \quad (8.3)$$

Using the Brézis and Lieb Lemma as in [4] we get

$$\|Tf_n^R - Tf^R\|_6^6 = \|Tf_n^R\|_6^6 - \|Tf^R\|_6^6 + o_{n,R}(1), \quad (8.4)$$

where  $o_{n,R}(1) \rightarrow 0$  as  $n \rightarrow \infty$ , when we keep  $R$  fixed. Using (8.1), (8.2) and (8.4) we obtain

$$\|Tf_n - Tf\|_6^6 - (\|Tf_n\|_6^6 - \|Tf\|_6^6) \leq o_{n,R}(1) + C\Theta(R), \quad (8.5)$$

By the weak convergence

$$\|f_n - f\|_2^2 = \|f_n\|_2^2 - \|f\|_2^2 + o_n(1) \quad (8.6)$$

or equivalently

$$\lim_{n \rightarrow \infty} \|f_n - f\|_2^2 = 1 - \|f\|_2^2. \quad (8.7)$$

Using that  $\{f_n\}_{n \in \mathbb{N}}$  is a maximizing sequence for  $T$  we get

$$\|T\|^2 = \frac{\|Tf_n\|_6^2}{\|f_n\|_2^2} + o_n(1) \leq \frac{(\|Tf_n - Tf\|_6^6 + \|Tf\|_6^6 + o_{n,R}(1) + C\Theta(R))^{1/3}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1) \quad (8.8)$$

$$\leq \frac{\|Tf_n - Tf\|_6^2 + \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1), \quad (8.9)$$

where we used the inequality

$$(a + b + c)^t \leq a^t + b^t + c^t, \text{ for all } a, b, c \geq 0 \text{ and } 0 \leq t \leq 1.$$

The continuity of  $T$  and (8.9) imply

$$\|T\|^2 \leq \frac{\|T\|^2 \|f_n - f\|_2^2 + \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1),$$

and hence

$$\|T\|^2(\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)) \leq \|T\|^2\|f_n - f\|_2^2 + \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}},$$

that after canceling terms implies

$$\|T\|^2(\|f\|_2^2 + o_n(1)) \leq \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}.$$

Since we also have the inequality  $\|Tf\|_6^2 \leq \|T\|^2\|f\|_2^2$  we can take the limit as  $n \rightarrow \infty$  in the previous inequality followed by a limit as  $R \rightarrow \infty$  to obtain

$$\|Tf\|_6 = \|T\|\|f\|_2. \quad (8.10)$$

Note that this implies that  $f$  is an extremizer for  $T$  since  $f \neq 0$  by hypothesis. It remains to prove that  $f_n \rightarrow f$  in  $L^2(\Gamma^2)$ . Using (8.9) and (8.10) we get

$$\|T\|^2 \leq \frac{\|T\|^2\|f\|_2^2 + \|Tf_n - Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1),$$

which as before implies

$$\|T\|^2(\|f_n - f\|_2^2 + o_n(1)) \leq \|Tf_n - Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}. \quad (8.11)$$

and by continuity of  $T$

$$\|Tf_n - Tf\|_6^2 \leq \|T\|^2\|f_n - f\|_2^2. \quad (8.12)$$

Using (8.11), (8.12) and (8.7) we can take the limit as  $n \rightarrow \infty$  and then the limit as  $R \rightarrow \infty$  to obtain

$$\lim_{n \rightarrow \infty} \|Tf_n - Tf\|_6^2 = \|T\|^2(1 - \|f\|_2^2).$$

Now, using this last equality together with (8.5) and the fact that  $\{f_n\}_{n \in \mathbb{N}}$  is an  $L^2$ -normalized extremizing sequence gives

$$\|T\|^6 = \lim_{n \rightarrow \infty} \|Tf_n\|_6^6 = \|T\|^6(1 - \|f\|_2^2)^3 + \|T\|^6\|f\|_2^6,$$

therefore

$$(1 - \|f\|_2^2)^3 + \|f\|_2^6 = 1 \text{ and } \|f\|_2 \leq 1.$$

This easily implies that either  $\|f\|_2 = 1$  or  $\|f\|_2 = 0$ . The latter case does not hold since  $f \neq 0$  by assumption. Thus  $\|f\|_2 = 1$  and  $f_n \rightarrow f$  in  $L^2(\Gamma^2)$ .  $\square$

## 9. THE CHRIST-SHAO CONCENTRATION COMPACTNESS ARGUMENT

We are now ready to use the Christ-Shao concentration compactness argument to gain control over extremizing sequences. We follow the same lines as in [2] so many of the arguments are the same. We will indicate when changes are needed, but will not go over the entire argument. We note that there will be one part we will do in a different way, namely we will not study “cross terms”, section 14 in [2], but instead we will use an argument in [4] that to the author seems much easier. In this way we avoid the need of the use of Fourier integral operators (sections 7.3 and 7.4 in [2]).

We now state the results from [2] of interest for us. We indicate the changes needed in our case and if we just state the result without proof is because it is exactly as in [2] with the possible exception of changing norms from  $L^4$  in their case to  $L^6$  in our case.



**Definition 9.1.** A nonzero function  $f \in L^2(\Gamma^2)$  is said to be a  $\delta$ -nearly extremal for (1.2) if

$$\|Tf\|_{L^6(\mathbb{R}^3)} \geq (1 - \delta)C\|f\|_{L^2(\Gamma^2)}.$$

**Lemma 9.2.** Let  $f = g + h \in L^2(\Gamma^2)$ . Suppose that  $g \perp h$ ,  $g \neq 0$ , and that  $f$  is a  $\delta$ -nearly extremal for some  $\delta \in (0, \frac{1}{4}]$ . Then

$$\frac{\|h\|_2}{\|f\|_2} \leq C \max\left(\frac{\|Th\|_6}{\|h\|_2}, \delta^{1/2}\right). \quad (9.1)$$

Here  $C < \infty$  is a constant independent of  $g, h$ .

Let  $\mathcal{M}$  be the set of all caps modulo the equivalence relation  $\mathcal{C} \sim \mathcal{C}'$  if there exists  $k \in \mathbb{Z}$  such that  $\mathcal{C}, \mathcal{C}' \subseteq [2^{k-1}, 2^k] \times [0, 2\pi]$ . We define the following metric on  $\mathcal{M}$ .

**Definition 9.3.** For any two caps  $\mathcal{C}, \mathcal{C}' \subseteq \Gamma^2$ ,

$$\varrho([\mathcal{C}], [\mathcal{C}']) = |k - k'| \quad (9.2)$$

where  $\mathcal{C} = [2^{k-1}, 2^k] \times J$  and  $\mathcal{C}' = [2^{k'-1}, 2^{k'}] \times J'$  and  $[\mathcal{C}]$  denotes the equivalent class  $[\mathcal{C}] = \{[2^{k-1}, 2^k] \times I : I \subseteq [0, 2\pi] \text{ and } I \text{ is an interval}\}$ .

We will also write  $\varrho(\mathcal{C}, \mathcal{C}') = \varrho([\mathcal{C}], [\mathcal{C}'])$ .

The equivalent of [2, Lemma 7.5] is the bilinear estimate in Lemma 3.3. We restate it in the language of caps.

**Lemma 9.4.** Let  $f, g \in L^2(\Gamma^2)$  supported in the caps  $\mathcal{C}, \mathcal{C}'$  respectively, then

$$\|Tf \cdot Tg\|_{L^3} \leq C 2^{-\varrho(\mathcal{C}, \mathcal{C}')/6} \|f\|_2 \|g\|_2,$$

in particular

$$\|T\chi_{\mathcal{C}} \cdot T\chi_{\mathcal{C}'}\|_{L^3} \leq C 2^{-\varrho(\mathcal{C}, \mathcal{C}')/6} |\mathcal{C}|^{1/2} |\mathcal{C}'|^{1/2}.$$

Here  $C < \infty$  is a universal constant.

We now move to the decomposition algorithm, [2, Section 8]. Note that the decomposition algorithm does not depend on the specific manifold we are dealing with, it just requires Lemma 6.1.

Given a nonnegative function  $f \in L^2(\Gamma^2)$ , the decomposition algorithm gives, in brief, a sequence of disjoint caps  $\{\mathcal{C}_\nu\}_{\nu \in \mathbb{N}}$ , constants  $\{C_\nu\}_{\nu \in \mathbb{N}}$ , nonnegative functions  $f_\nu$  supported on  $\mathcal{C}_\nu$  and nonnegative functions  $G_\nu$ , whose support is disjoint from  $f_0 + \dots + f_{\nu-1}$ , such that  $f_\nu \leq C_\nu |\mathcal{C}_\nu|^{-1/2} \chi_{\mathcal{C}_\nu}$ ,  $f = \sum_{\nu=0}^{N-1} f_\nu + G_N$ , for all  $N \geq 0$ , and  $f = \sum_{\nu=0}^{\infty} f_\nu$ , where the sum is  $L^2(\Gamma^2)$ -convergent, [2, Lemma 8.1].

Other useful properties can be obtained if  $f$  is nearly extremal for (1.2). Lemmas 8.2, 8.3 and 8.4 in [2] have exact analogs for the cone. We mention here the ones we will use.

The analog of Lemma 8.3 in [2] for the cone implies

**Lemma 9.5.** There exists a sequence of positive constants  $\gamma_\nu \rightarrow 0$  and a function  $N : (0, \frac{1}{2}] \rightarrow \mathbb{Z}^+$  satisfying  $N(\delta) \rightarrow \infty$  as  $\delta \rightarrow 0$  such that for any nonnegative  $f \in L^2(\Gamma^2)$  that is  $\delta$ -nearly extremal, the functions  $\{f_\nu, G_\nu\}_{\nu \in \mathbb{N}}$  obtained from the decomposition algorithm satisfy

$$\|G_\nu\|_2, \|f_\nu\|_2 \leq \gamma_\nu \|f\|_2 \text{ for all } \nu \leq N(\delta).$$

This lemma will be used in the following way: given  $\varepsilon > 0$  we can find  $\nu(\varepsilon)$  such that  $\gamma_\nu < \varepsilon^3$  for all  $\nu \geq \nu(\varepsilon)$ . If we let  $\delta(\varepsilon)$  be such that  $N(\delta) \geq \nu_0$  for all  $\delta \leq \delta(\varepsilon)$  it follows that an inequality  $\|G_N\|_2 \geq \varepsilon^3$  applied to a  $\frac{1}{2}\delta(\varepsilon)$ -nearly extremal  $f$  whose decomposition is  $\{f_\nu, G_\nu\}_{\nu \in \mathbb{N}}$ , implies  $N \leq N(\delta(\varepsilon))$  or  $N > N(\frac{1}{2}\delta(\varepsilon))$ . The fact that  $\{\|G_\nu\|_2\}_{\nu \in \mathbb{N}}$  is a nonincreasing sequence discards the second possibility, hence  $N \leq N(\delta(\varepsilon)) =: M_\varepsilon$ .

From Lemma 6.1, the analog of Lemma 8.4 in [2] follows

**Lemma 9.6.** *For any  $\varepsilon > 0$  there exist  $\delta_\varepsilon > 0$  and  $C_\varepsilon < \infty$  such that for every  $\delta_\varepsilon$ -nearly extremal nonnegative function  $f \in L^2(\Gamma^2)$ , the functions  $f_\nu, G_\nu$  and the caps  $\mathcal{C}_\nu$  associated to  $f$  via the decomposition algorithm satisfy  $f_\nu \leq C_\varepsilon \|f\|_2 |\mathcal{C}_\nu|^{-1/2} \chi_{\mathcal{C}_\nu}$  and  $\|f_\nu\|_2 \geq \delta_\varepsilon \|f\|_2$  whenever  $\|G_\nu\|_2 \geq \varepsilon \|f\|_2$ .*

We now move to the analog of [2, Lemma 9.2]. The only difference in the proof compared to that in the paper of Christ-Shao is that we need to replace the  $L^4$  norm by the  $L^6$  norm.

**Lemma 9.7.** *For any  $\varepsilon > 0$  there exists  $\delta > 0$  and  $\lambda < \infty$  such that for any  $0 \leq f \in L^2(\Gamma^2)$  which is  $\delta$ -nearly extremal, the summands  $f_\nu$  produced by the decomposition algorithm and the associated caps  $\mathcal{C}_\nu$  satisfy*

$$\varrho(\mathcal{C}_j, \mathcal{C}_k) \leq \lambda \text{ whenever } \|f_j\|_2 \geq \varepsilon \|f\|_2 \text{ and } \|f_k\|_2 \geq \varepsilon \|f\|_2. \quad (9.3)$$

*Proof.* It suffices to prove this for all sufficiently small  $\varepsilon$ . Let  $f$  be a nonnegative  $L^2$  function which satisfies  $\|f\|_2 = 1$  and is  $\delta$ -nearly extremal for a sufficiently small  $\delta = \delta(\varepsilon)$ , and let  $\{G_\nu, f_\nu\}_{\nu \in \mathbb{N}}$  be associated to  $f$  via the decomposition algorithm. Set  $F = \sum_{\nu=0}^N f_\nu$ .

Suppose that  $\|f_{j_0}\|_2 \geq \varepsilon$  and  $\|f_{k_0}\|_2 \geq \varepsilon$ . Let  $N$  be the smallest integer such that  $\|G_{N+1}\|_2 < \varepsilon^3$ . Since  $\|G_\nu\|_2$  is a nonincreasing function of  $\nu$ , and since  $\|f_\nu\|_2 \leq \|G_\nu\|_2$ , necessarily  $j_0, k_0 \leq N$ . Moreover, from the comment after Lemma 9.5, there exists  $M_\varepsilon < \infty$  depending only on  $\varepsilon$  such that  $N \leq M_\varepsilon$ . By Lemma 9.6, if  $\delta$  is chosen to be a sufficiently small function of  $\varepsilon$  then since  $\|G_\nu\|_2 \geq \varepsilon^3$  for all  $\nu \leq N$ ,  $f_\nu \leq \theta(\varepsilon) |\mathcal{C}_\nu|^{-1/2} \chi_{\mathcal{C}_\nu}$  for all such  $\nu$ , where  $\theta$  is a continuous, strictly positive function on  $(0, 1]$ .

Now let  $\lambda < \infty$  be a large quantity to be specified. It suffices to show that if  $\delta(\varepsilon)$  is sufficiently small, an assumption that  $\varrho(\mathcal{C}_j, \mathcal{C}_k) > \lambda$  implies an upper bound, which depends only on  $\varepsilon$ , for  $\lambda$ .

There exists a decomposition  $F = F_1 + F_2 = \sum_{\nu \in S_1} f_\nu + \sum_{\nu \in S_2} f_\nu$  where  $[0, N] = S_1 \cup S_2$  is a partition of  $[0, N]$ ,  $j_0 \in S_1$ ,  $k_0 \in S_2$ , and  $\varrho(\mathcal{C}_j, \mathcal{C}_k) \geq \lambda/2N \geq \lambda/2M_\varepsilon$  for all  $j \in S_1$  and  $k \in S_2$ . Certainly  $\|F_1\|_2 \geq \|f_{j_0}\|_2 \geq \varepsilon$  and similarly  $\|F_2\|_2 \geq \varepsilon$ . The cross term satisfies

$$\|TF_1 \cdot TF_2\|_3 \leq \sum_{j \in S_1} \sum_{k \in S_2} \|Tf_j \cdot Tf_k\|_3 \leq M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon) \theta(\varepsilon)^2,$$

where  $\gamma(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$  by Lemma 3.3. Expand

$$\begin{aligned} \|TF \cdot TF\|_3^3 &\leq \|TF_1\|_6^6 + \|TF_2\|_6^6 + 15\|(TF_1)^2 \cdot TF_2\|_2^2 + 15\|TF_1 \cdot (TF_2)^2\|_2^2 \\ &\quad + 20\|TF_1 \cdot TF_2\|_3^3 + 6\|(TF_1)^5 \cdot TF_2\|_1 + 6\|TF_1 \cdot (TF_2)^5\|_1. \end{aligned}$$

By using Hölder's inequality

$$\begin{aligned} \|TF \cdot TF\|_3^3 &\leq \|TF_1\|_6^6 + \|TF_2\|_6^6 + 20\|TF_1 \cdot TF_2\|_3^3 \\ &\quad + 6(\|TF_1\|_6^4 + \|TF_2\|_6^4)\|TF_1 \cdot TF_2\|_3 \\ &\quad + 15(\|TF_1\|_6^2 + \|TF_2\|_6^2)\|TF_1 \cdot TF_2\|_3^2, \end{aligned}$$

and using that  $T$  is continuous and denoting  $\mathbf{C} = \|T\|$  we get

$$\begin{aligned} \|TF \cdot TF\|_3^3 &\leq \mathbf{C}^6(\|F_1\|_2^6 + \|F_2\|_2^6) + 20\|TF_1 \cdot TF_2\|_3^3 \\ &\quad + 6\mathbf{C}^4(\|F_1\|_2^4 + \|F_2\|_2^4)\|TF_1 \cdot TF_2\|_3 \\ &\quad + 15\mathbf{C}^2(\|F_1\|_2^2 + \|F_2\|_2^2)\|TF_1 \cdot TF_2\|_3^2, \end{aligned}$$

Since  $F_1$  and  $F_2$  have disjoint supports,  $\|F_1\|_2^2 + \|F_2\|_2^2 \leq \|f\|_2 = 1$  and consequently

$$\begin{aligned} \|F_1\|_2^4 + \|F_2\|_2^4 &\leq \max(\|F_1\|_2^2, \|F_2\|_2^2) \cdot (\|F_1\|_2^2 + \|F_2\|_2^2) \leq (1 - \varepsilon^2) \cdot 1 = 1 - \varepsilon^2, \\ \|F_1\|_2^6 + \|F_2\|_2^6 &\leq \max(\|F_1\|_2^2, \|F_2\|_2^2)^2 \cdot (\|F_1\|_2^2 + \|F_2\|_2^2) \leq (1 - \varepsilon^2)^2 \cdot 1 = (1 - \varepsilon^2)^2. \end{aligned}$$

Thus

$$\begin{aligned} \|TF \cdot TF\|_3^3 &\leq \mathbf{C}^6(1 - \varepsilon^2)^2 + 20(M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon) \theta(\varepsilon)^2)^3 \\ &\quad + 6\mathbf{C}^4(1 - \varepsilon^2)M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon) \theta(\varepsilon)^2 \\ &\quad + 15\mathbf{C}^2(M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon) \theta(\varepsilon)^2)^2. \end{aligned}$$

On the other hand repeating the previous calculations with  $F_1 = F$  and  $F_2 = f - F$  and using that  $\|f\|_2 = 1$ ,  $\|f - F\|_2 \leq \varepsilon^3 < 1$  we get

$$\begin{aligned} (1 - \delta)^6 \mathbf{C}^6 &\leq \|Tf \cdot Tf\|_3^3 \leq \|TF \cdot TF\|_3^3 + C\|f\|_2\|f - F\|_2 \\ &\leq \|TF \cdot TF\|_3^3 + C\varepsilon^3. \end{aligned}$$

Hence

$$\begin{aligned} (1 - \delta)^6 \mathbf{C}^6 &\leq C\varepsilon^3 + \mathbf{C}^6(1 - \varepsilon^2)^2 + 20(M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon) \theta(\varepsilon)^2)^3 \\ &\quad + 6\mathbf{C}^4(1 - \varepsilon^2)M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon) \theta(\varepsilon)^2 + 15\mathbf{C}^2(M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon) \theta(\varepsilon)^2)^2. \end{aligned}$$

Since  $\gamma(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for all sufficiently small  $\varepsilon > 0$  this implies an upper bound, which depends on  $\varepsilon$ , for  $\lambda$ , as was to be proved.  $\square$

**Proposition 9.8.** *There exists a function  $\Theta : [1, \infty) \rightarrow (0, \infty)$  satisfying  $\Theta(R) \rightarrow 0$  as  $R \rightarrow \infty$  with the following property. For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that any nonnegative function  $f \in L^2(\Gamma^2)$  satisfying  $\|f\|_2 = 1$  which is  $\delta$ -nearly extremal may be decomposed as  $f = F + G$  where  $F, G$  are nonnegative with disjoint supports,  $\|G\|_2 < \varepsilon$ , and there exists  $k \in \mathbb{Z}$  such that*

$$\int_{|y| < 2^k R^{-1}} |F(y)|^2 d\sigma(y) + \int_{|y| > 2^k R} |F(y)|^2 d\sigma(y) \leq \Theta(R), \quad \forall R \geq 1. \quad (9.4)$$

**Remark 9.9.** It will be clear from the proof of Lemma 9.10 that if there exists a cap  $\mathcal{C} \subset [2^{k_0}, 2^{k_0+1}] \times [0, 2\pi]$  such that  $g := f\chi_{\mathcal{C}}$  satisfies

$$\begin{aligned} |g(x)| &\leq C\|f\|_2|\mathcal{C}|^{-1/2}\chi_{\mathcal{C}}, \forall x \text{ and} \\ \|g\|_2 &\geq c\|f\|_2 \end{aligned}$$

for  $C, c$  universal constants, then we can take  $k = k_0$ , and  $F \geq g$  on  $\Gamma^2$ .

**Lemma 9.10.** *There exists a function  $\Theta : [1, \infty) \rightarrow (0, \infty)$  satisfying  $\Theta(R) \rightarrow 0$  as  $R \rightarrow \infty$  with the following property. For any  $\varepsilon > 0$  and  $\bar{R} \in [1, \infty)$  there exists  $\delta > 0$  such that any nonnegative function  $f \in L^2(\Gamma^2)$  satisfying  $\|f\|_2 = 1$  which is  $\delta$ -nearly extremal may be decomposed as  $f = F + G$  where  $F, G$  are nonnegative with disjoint supports,  $\|G\|_2 < \varepsilon$ , and there exists  $k \in \mathbb{Z}$  such that for any  $R \in [1, \bar{R}]$*

$$\int_{|y| < 2^k R^{-1}} |F(y)|^2 d\sigma(y) + \int_{|y| > 2^k R} |F(y)|^2 d\sigma(y) \leq \Theta(R). \quad (9.5)$$

*Proof that Lemma 9.10 implies Proposition 9.8.* Let  $\Theta$  be the function promised by the lemma. Let  $\varepsilon, f$  be given, and assume without loss of generality that  $\varepsilon$  is small. Assuming as we may that  $\Theta$  is a continuous, strictly decreasing function, define  $\bar{R} = \bar{R}(\varepsilon)$  by the equation  $\Theta(\bar{R}) = \varepsilon^2/2$ . Let  $k, \delta = \delta(\varepsilon, \bar{R}(\varepsilon))$  along with  $F, G$  satisfy the conclusions of the lemma relative to  $\varepsilon, \bar{R}(\varepsilon)$ . Define  $\chi$  to be the characteristic function of the set of all  $x \in \mathbb{R}^2$  which satisfy  $|x| > 2^k \bar{R}$  or  $|x| < 2^k \bar{R}^{-1}$ . Redecompose  $f = \tilde{F} + \tilde{G}$ , where  $\tilde{F} = (1 - \chi)F$  and  $\tilde{G} = G + \chi F$ . Then  $\|\tilde{G}\|_2 < 2\varepsilon$ , while  $\tilde{F}$  satisfies the required inequalities. For  $R > \bar{R}$  we have  $\tilde{F} = 0$ , and if  $R \leq \bar{R}$ , then,

$$\int_{|x| > 2^k R} |\tilde{F}(x)|^2 d\sigma(x) \leq \int_{|x| > 2^k R} |F(x)|^2 d\sigma(x) \leq \Theta(R),$$

and similarly for the other integral.  $\square$

We now prove Lemma 9.10, the analog of Lemma 10.1 of [2]

*Proof.* Let  $\eta : [1, \infty) \rightarrow (0, \infty)$  be a function to be chosen below, satisfying  $\eta(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This function will not depend on the quantity  $\bar{R}$ .

Let  $\bar{R} \geq 1$ ,  $R \in [1, \bar{R}]$ , and  $\varepsilon > 0$  be given. Let  $\delta = \delta(\varepsilon, \bar{R}) > 0$  be a small quantity to be chosen below. Let  $0 \leq f \in L^2(\Gamma^2)$  be a  $\delta$ -nearly extremal, with  $\|f\|_2 = 1$ .

Let  $\{f_\nu\}_{\nu \in \mathbb{N}}$  be the sequence of functions obtained by applying the decomposition algorithm to  $f$ . Choose  $\delta = \delta(\varepsilon) > 0$  sufficiently small and  $M = M(\varepsilon)$  sufficiently large to guarantee  $\|G_{M+1}\|_2 < \varepsilon/2$  and that  $f_\nu, G_\nu$  satisfy the conclusions of the analog of Lemma 8.4 and Lemma 8.3 in [2] for  $\nu \leq M$ . Set  $F = \sum_{\nu=0}^M f_\nu$ . Then  $\|f - F\|_2 = \|G_{M+1}\|_2 < \varepsilon/2$ .

Let  $N \in \{0, 1, 2, \dots\}$  be the minimum of  $M$ , and the smallest number such that  $\|f_{N+1}\|_2 < \eta$ .  $N$  is bounded above by a quantity which depends only on  $\eta$ . Set  $\mathcal{F} = \mathcal{F}_N = \sum_{k=0}^N f_\nu$ . It follows from Lemma 8.4 in [2], that

$$\|F - \mathcal{F}\|_2 < \gamma(\eta) \text{ where } \gamma(\eta) \rightarrow 0 \text{ as } \eta \rightarrow 0. \quad (9.6)$$

This function  $\gamma$  is independent of  $\varepsilon, \bar{R}$ .

To prove the lemma, we must produce an integer  $k$  and must establish the existence of  $\Theta$ . To do the former is simple: To  $f_0$  is associated a cap  $\mathcal{C}_0 \subset [2^{k_0-1}, 2^{k_0}] \times [0, 2\pi]$  such that  $f_0 \leq C|\mathcal{C}_0|^{-1/2}\chi_{\mathcal{C}_0}$ , for some universal constant  $C$ .  $k = k_0$  is the required integer. Note that by Lemma 6.1,  $\|f_0\|_2 \geq c$  for some positive universal constant  $c$ . This implies, by Lemma 6.2, that  $\|f_0\|_1 \geq c'|\mathcal{C}_0|^{1/2}$ , for some universal constant  $c'$ . This last remark will be of use after rescaling.

Suppose that functions  $R \mapsto \eta(R)$  and  $R \mapsto \Theta(R)$  are chosen so that

$$\begin{aligned} \eta(R) &\rightarrow 0 \text{ as } R \rightarrow \infty, \\ \gamma(\eta(R)) &\leq \Theta(R)^{\frac{1}{2}} \text{ for all } R. \end{aligned}$$

Then by (9.6),  $F - \mathcal{F}$  already satisfies the desired inequalities in  $L^2(\Gamma^2)$ , so it suffices to show that  $\mathcal{F}(x) \equiv 0$  whenever  $|x| \geq 2^{k_0}R$  or  $|x| \leq 2^{k_0}R^{-1}$ .

Each summand satisfies  $f_k \leq C(\eta)|\mathcal{C}_k|^{-1/2}\chi_{\mathcal{C}_k}$ , where  $C(\eta) < \infty$  depends only on  $\eta$ , and in particular,  $f_k$  is supported in  $\mathcal{C}_k$ .  $\|f_k\|_2 \geq \eta$  for all  $k \leq N$ , by definition of  $N$ . Therefore by Lemma 9.7, there exists a function  $\eta \mapsto \lambda(\eta) < \infty$  such that if  $\delta$  is sufficiently small as a function of  $\eta$  then  $\varrho(\mathcal{C}_k, \mathcal{C}_0) \leq \lambda(\eta)$  for all  $k \leq N$ . This is needed for  $\eta = \eta(R)$  for all  $R$  in the compact set  $[1, \bar{R}]$ , so such a  $\delta$  may be chosen as a function of  $\bar{R}$  alone; conditions already imposed on  $\delta$  above make it a function of both  $\varepsilon, \bar{R}$ .

Let  $\tau_k \in \mathbb{Z}$  be such that  $\mathcal{C}_k \subset [2^{\tau_k}, 2^{\tau_k+1}] \times [0, 2\pi]$ . Then  $|\tau_k - k_0| \leq \lambda(\eta)$ , so  $2^{\tau_k} \leq 2^{k_0}2^{\lambda(\eta)}$  and  $2^{\tau_k} \geq 2^{k_0}2^{-\lambda(\eta)}$ . Choosing  $R \mapsto \eta(R)$  so that  $2^{\lambda(\eta(R))} \leq R$  gives  $\mathcal{F}(x) \equiv 0$  when  $|x| \geq 2^{k_0}R$  and when  $|x| \leq 2^{k_0}R^{-1}$ .

We therefore choose a function  $\eta$  such that  $\eta(R) \rightarrow 0$  as  $R \rightarrow \infty$  slow enough to ensure  $\lambda(\eta(R)) \leq \log(R)$ . We then choose  $\Theta(R) = \gamma(\eta(R))^2$ , for all  $R \geq 0$ .  $\square$

We are now ready to give a proof of Theorem 8.2 based in the Christ-Shao concentration compactness argument.

*Alternative proof of Theorem 8.2.* Let  $\{f_n\}_{n \in \mathbb{N}}$  be an extremizing sequence. We start as in the proof of Theorem 8.2 by using Lemma 6.1 with  $\delta = 1/2$  to decompose  $f_n = g_n + h_n$  and to obtain a cap  $\mathcal{C}_n$  satisfying the conclusions of Lemma 6.1. We then find  $\{\lambda_n\}_{n \in \mathbb{N}}$ ,  $\{t_n\}_{n \in \mathbb{N}}$  and  $\{\theta_n\}_{n \in \mathbb{N}}$  such that the set  $L_{t_n}^{-1}R_{\theta_n}^{-1}D_{\lambda_n}^{-1}\mathcal{C}_n$  is contained in a bounded region independent of  $n$  and has measure comparable to 1.

Define  $\tilde{f}_n = L_{t_n}^*R_{\theta_n}^*D_{\lambda_n}^*f_n$ ,  $\tilde{g}_n = L_{t_n}^*R_{\theta_n}^*D_{\lambda_n}^*g_n$ ,  $\tilde{h}_n = L_{t_n}^*R_{\theta_n}^*D_{\lambda_n}^*h_n$  and  $\tilde{\mathcal{C}}_n = L_{t_n}^{-1}R_{\theta_n}^{-1}D_{\lambda_n}^{-1}\mathcal{C}_n$ . Then  $\tilde{g}_n$  and  $\tilde{h}_n$  have disjoint supports,  $\tilde{g}_n$  is supported on  $\tilde{\mathcal{C}}_n \subset [\frac{1}{4}, 1] \times [0, 2\pi]$ ,  $\sigma(\tilde{\mathcal{C}}_n) \geq \frac{1}{2}$  and there exist  $0 < c, C < \infty$  independent of  $n$  such that

$$|\tilde{g}_n(x)| \leq C\|\tilde{f}_n\|_2|\tilde{\mathcal{C}}_n|^{-1/2}\chi_{\tilde{\mathcal{C}}_n}(x), \quad \text{and} \quad \|\tilde{g}_n\|_2 \geq c\|\tilde{f}_n\|_2. \quad (9.7)$$

This implies that  $\tilde{f}_n$  satisfies

$$\|\tilde{f}_n\chi_{\{\frac{1}{4} \leq |y| \leq 1\}}\|_2 > c' \quad \text{and} \quad \int \tilde{f}_n\chi_{\{\frac{1}{4} \leq |y| \leq 1\}}d\sigma(y) > c', \quad (9.8)$$

for all  $n$  with a constant  $c' > 0$  independent of  $n$ .

We now apply Proposition 9.8 to  $\{\tilde{f}_n/\|\tilde{f}_n\|_2\}_{n \in \mathbb{N}}$  with  $\varepsilon_n = 1/n$ ,  $n \geq 1$ , to obtain a subsequence of  $\{\tilde{f}_n\}_{n \in \mathbb{N}}$  (that we call the same), that satisfies the following. Each  $\tilde{f}_n$

can be decomposed as  $\tilde{f}_n = F_n + G_n$ , with  $F_n, G_n$  nonnegative with disjoint supports,  $\|G_n\|_2 < \frac{1}{n}$  and  $F_n$  satisfies (9.4) for certain  $k = k_n \in \mathbb{Z}$ .

We see that as  $\{\tilde{f}_n\}_{n \in \mathbb{N}}$  is an extremizing sequence of nonnegative functions for (1.2) so is  $\{F_n\}_{n \in \mathbb{N}}$  and we claim it satisfies the hypotheses of Proposition 8.4, after passing to a subsequence if necessary.

From (9.8) it follows that for all  $n$  large enough  $F_n$  satisfies

$$\|F_n \chi_{\{\frac{1}{4} \leq |y| \leq 1\}}\|_2 > \frac{c'}{2} \quad \text{and} \quad \int F_n \chi_{\{\frac{1}{4} \leq |y| \leq 1\}} d\sigma(y) > \frac{c'}{2}. \quad (9.9)$$

The first inequality in (9.9) together with the  $L^2$ -decay estimate (9.4) imply that  $\{k_n\}_{n \in \mathbb{N}}$  is a bounded sequence. After passing to a subsequence,  $F_n \rightharpoonup F$  for some  $F \in L^2(\Gamma^2)$  and  $F \neq 0$  since the  $F_n$ 's satisfy the second inequality in (9.9). Therefore  $\{F_n\}_{n \in \mathbb{N}}$  satisfies all the hypotheses of Proposition 8.4 and thus  $F_n \rightarrow F$  in  $L^2(\Gamma^2)$ . Therefore  $\tilde{f}_n \rightarrow F$  in  $L^2(\Gamma^2)$  which shows that  $\{f_n\}_{n \in \mathbb{N}}$  is precompact up to symmetries of the cone as needed.  $\square$

## 10. ON CONVERGENCE OF EXTREMIZING SEQUENCES

In this section we prove Theorem 1.4. We start with a general discussion.

Let  $(X, \mathcal{B}, \mu)$  be a measure space and let  $G$  be a group acting on  $L^2(X)$ , with an action that preserves the  $L^2$  norm, that is  $\|g^*f\|_{L^2(X)} = \|f\|_{L^2(X)}$  for all  $g \in G$  and  $f \in L^2(X)$ . For an element  $f \in L^2(X)$  we consider its orbit under  $G$ ,  $G(f) := \{g^*f : g \in G\}$ .

**Proposition 10.1.** *Let  $f \in L^2(X)$  and  $\{f_n\}_{n \in \mathbb{N}}$  a sequence in  $L^2(X)$  with the property that every subsequence has an  $L^2$ -convergent subsequence whose limit lies on  $G(f)$ . Then there exists a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset G$  such that  $g_n^*f_n \rightarrow f$  in  $L^2(X)$ , as  $n \rightarrow \infty$ .*

*Proof.* For each  $n$  let  $g_n \in G$  be such that

$$\|g_n^*f_n - f\|_{L^2(X)} \leq \inf_{g \in G} \|g^*f_n - f\|_{L^2(X)} + \frac{1}{n}.$$

We show that  $\{g_n^*f_n\}_{n \in \mathbb{N}}$  converges to  $f$  by showing that every subsequence has a further subsequence that converges to  $f$ . Take a subsequence (that we call the same),  $\{g_n^*f_n\}_{n \in \mathbb{N}}$ . By hypothesis,  $f_n$  has a convergent subsequence (that we call the same) to an element in  $G(f)$ . That is  $f_n \rightarrow g^*f$ , as  $n \rightarrow \infty$ , for some  $g \in G$ . By the definition of  $g_n$  and the invariance of the norm under the action of  $G$  we get

$$\|g_n^*f_n - f\|_{L^2(X)} \leq \|(g^{-1})^*f_n - f\|_{L^2(X)} + \frac{1}{n} = \|f_n - g^*f\|_{L^2(X)} + \frac{1}{n} \rightarrow 0$$

as  $n \rightarrow \infty$ .  $\square$

From Theorem 1.5 the extremizers for (1.2) are all of the form

$$g(x_1, x_2, x_3) = e^{-ax_3 - bx_2 - cx_1 + d}, \quad (10.1)$$

where  $a, b, c, d \in \mathbb{C}$  and  $|(\Re b, \Re c)| < \Re a$ , and here  $x_3 = \sqrt{x_1^2 + x_2^2}$ . As indicated in [6], any extremizer can be obtained from  $g_0(x_1, x_2, x_3) = e^{-x_3}$  by applying Lorentz transformations and dilations.

We define  $G$  as the group generated by  $L^t, M^s$  and  $D_r$ ,  $s, t \in (-1, 1)$ ,  $r > 0$  under composition. The action of  $G$  is given by the action of the generators as in (7.1) :  $L^{t*}f = f \circ L^t$ ,  $M^{s*}f = f \circ M^s$  and  $D_r^*f = r^{1/2}f \circ D^r$ . That  $G$  preserves the  $L^2(\Gamma^2)$  norm follows from the Lorentz invariance of  $\sigma$ .

**Lemma 10.2.** *The set of real,  $L^2$ -normalized extremizers for inequality (1.2) equals the orbit of  $g_0(y) = \pi^{-1/2}e^{-|y|}$ ,  $y \in \mathbb{R}^2$ , under the group  $G$ .*

*Proof.* A computation shows

$$L^t \circ M^s(x_1, x_2, x_3) = \left( \frac{x_1 + t \frac{x_3 + sx_2}{(1-s^2)^{1/2}}}{(1-t^2)^{1/2}}, \frac{x_2 + sx_3}{(1-s^2)^{1/2}}, \frac{\frac{x_3 + sx_2}{(1-s^2)^{1/2}} + tx_1}{(1-t^2)^{1/2}} \right).$$

Then

$$g_0 \circ L^t \circ M^s \circ D_r = r^{\frac{1}{2}} \pi^{-\frac{1}{2}} \exp \left( -\frac{rx_3}{(1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}} - \frac{sr x_2}{(1-s^2)^{\frac{1}{2}}(1-t^2)^{\frac{1}{2}}} - \frac{tr x_1}{(1-t^2)^{\frac{1}{2}}} \right).$$

For given  $a > 0$  and  $b, c \in \mathbb{R}$  with  $|(b, c)| < a$  we want to solve the equations

$$\begin{aligned} \frac{r}{(1-s^2)^{1/2}(1-t^2)^{1/2}} &= a, \\ \frac{sr}{(1-s^2)^{1/2}(1-t^2)^{1/2}} &= b, \\ \frac{tr}{(1-t^2)^{1/2}} &= c. \end{aligned}$$

Since  $a \neq 0$  and  $|b| < a$  we have  $b/a = s \in (-1, 1)$ . Also  $c/a = t(1-s^2)^{1/2}$ , so  $t = \frac{c}{a(1-s^2)^{1/2}} = \frac{c}{(a^2-b^2)^{1/2}}$  and we see that  $|t| < 1$ . Finally  $r = a(1-s^2)^{1/2}(1-t^2)^{1/2} = (a^2 - b^2 - c^2)^{1/2}$ . The  $L^2$ -norm is preserved by the action of  $G$  thus a normalized, real extremizer  $g(y) = e^{-a|y|-by_2-cy_1+d}$  can be obtained from  $g_0$  by composing with  $L^t \circ M^s \circ D_r$  for the computed values of  $t, s$  and  $r$ .  $\square$

*Proof of Theorem 1.4.* From the previous discussion we have that the group  $G$  gives all real extremizers as the orbit of  $g_0$ . Proposition 10.1, Theorem 1.3 and Theorem 1.5 give a proof of Theorem 1.4.  $\square$

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## REFERENCES

- [1] Emanuel Carneiro, *A sharp inequality for the Strichartz norm*, Int. Math. Res. Not. **16** (2009), 3127–3145.
- [2] Michael Christ and Shuanglin Shao, *Existence of extremizers for a Fourier restriction inequality*, Preprint (2010). To appear in Anal. PDE.
- [3] Athanase Cotisolis and Nikolaos K. Tavoularis, *Best constants for Sobolev inequalities for higher order fractional derivatives*, J. Math. Anal. Appl. **295** (2004), no. 1, 225–236.



- [4] Luca Fanelli, Luis Vega, and Nicola Visciglia, *On the existence of maximizers for a family of restriction theorems*, Bull. London Math. Soc. **43** (2011), no. 4, 811–817.
- [5] ———, *Existence of maximizers for Sobolev-Strichartz inequalities*, Adv. Math. **229** (2012), no. 3, 1912–1923.
- [6] Damiano Foschi, *Maximizers for the Strichartz inequality*, J. Eur. Math. Soc. **9** (2007), no. 4, 739–774.
- [7] A. Moyua, A. Vargas, and L. Vega, *Restriction theorems and maximal operators related to oscillatory integrals in  $\mathbb{R}^3$* , Duke Math. J. **96** (1999), no. 3, 547–574.
- [8] Richard O’Neil, *Convolution operators and  $L(p, q)$  spaces*, Duke Math. J. **30** (1963), 129–142.
- [9] Michael Reed and Barry Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.
- [10] Christopher D. Sogge, *Lectures on nonlinear wave equations*, Monographs in Analysis, II, International Press, Boston, MA, 1995.
- [11] Elias M. Stein and Guido Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
- [12] Robert S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714.

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